Generalized Reliability Methods
for Systems with Standbys

By

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TO MY FATHER,

LEO LINTON

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GENERALIZED RELIABILITY METHODS FOR SYSTEMS WITH STANDBYS

By

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For repairable systems with standbys, several candidates for measuring reliability are (1) the time to system failure, (2) the number of repairs completed before failure and (3) the total time spent on repair before failure. In this research, some general procedures are developed for studying these and other reliability characteristics of systems with standbys. Emphasis is placed on a class of repairable systems known as k-out-of-n systems, $1 \leq k \leq n$, where n units in parallel redundancy are serviced by a single repairman and system failure occurs when k units are simultaneously inoperable for the first time.

Assuming λ_i and $F_i(\cdot)$ are, respectively, the failure rate and general repair distribution for unit i, the supplementary variable technique is used to find the transform of the time to system failure distribution for the k-out-of-n system. Using the principle of regeneration, an alternative derivation is also obtained for the 2-out-of-n system.

A conditional transform approach is applied to the 2-out-of-n system and the two-unit standby redundant system. For each system, transforms of distributions are obtained for the time to system failure, the idle time of the repairman and the time spent on repair. In the case of exponential failure and Erlangian repair capabilities, the generating function for the distribution of the number of renewals occurring during the life of each system is found.

Numerical results are presented for the 2-out-of-3 and 3-out-of-4 systems for several different repair disciplines. Numerical methods for evaluating moments of distributions expressed in terms of complex integrals are also discussed. Although specific results are not derived, the value of stochastic integrals for investigating attributes of redundant systems is clearly demonstrated.

CHAPTER 1

AN OVERVIEW

Introduction

High speed computers, sophisticated weaponry and faster means of communication and transportation have focused attention in the scientific community on the design and manufacture of systems which are both proficient and reliable. Most engineers would agree, however, that unless reliability is incorporated in the design phase, the final product will suffer regardless of corrective maintenance policy. Thus, it is the ability to measure reliability that concerns the engineer and designer of today.

For repairable systems with standbys, several candidates for measuring reliability are (1) the time to system failure, (2) the number of repairs completed before failure and (3) the total time spent on repair before failure. In this dissertation, some general procedures are developed for studying these and other reliability characteristics of systems with standbys. Emphasis is placed on a class of repairable systems known as k-out-of-n systems, $1 \leq k \leq n$, where n units (or machines) in parallel redundancy are serviced by a single repairman and system failure occurs when k units are simultaneously inoperable for the first time. The often referred to series and parallel arrangements are the special cases $k = 1$ and $k = n$, respectively. Because k-out-of-n systems (or combinations thereof) model a large class of redundant systems, this research encompasses a study of the attributes of such systems.

Previous Research Results

The body of methods and ideas used in studying characteristics related to the life of a system is referred to as mathematical reliability theory. Although a relatively new area of study, reliability theory has already been applied to problems involving power systems [1], [2], [3], [4], [5] and the regulation of traffic [6], [7], as well as to multicomponent structures [8], [9], [10], [11] and [12].

Takacs [13] in 1957 and Morse [14] in 1958 discussed properties of a system where several machines were serviced by a single repairman, and each machine was subject to exponential failure. Limiting distributions for the number of machines working were derived by Takacs for general repair and by Morse for exponential repair.

One of the earliest to treat first passage times [15] in the context of reliability theory was Gaver [16] in 1960. For a birth-death process with constant transition rates, Gaver derived a formula for $E[T_{ij}]$, $j > i$, where T_{ij} is the first passage time to go from state i to state j. In 1962, Belyayev [17] found the mean time to system failure (MTSF) for a system composed of two units in parallel redundancy, both of which are operating initially. Belyayev assumed that failures were exponential, repair was general, and the system failed when both units were inoperable for the first time. In 1963, Gaver [18] solved the same problem considered by Belyayev but used a modified approach. Both Belyayev and Gaver used the Cox [19] method of supplementary variables to analyze the process. Later in 1963, McGregor [20] derived formulas for the MTSF for the more general k-out-of-n system where both failures and repairs were exponential.

Thiruvengadam and Jaiswal [21] in 1964 and Jaiswal [22] in 1968 used discrete transforms to investigate the k-out-of-n system with general repair capabilities. In addition to several other system attributes, Thiruvengadam and Jaiswal found the distribution of a busy period generated by i units, but they did not consider properties of the time to system failure.

In 1966, Muth [23], Htun [24] and Srinivasan [25] discussed a standby redundant system composed of two units. Muth found the MTSF when the repairman has the capability of completing n repairs. As $n \to \infty$, Muth showed that the MTSF agreed with equation (20) in Gaver [18]. Htun assumed repair and failure rates were constant but different for each unit and derived the MTSF. Srinivasan assumed general failure and repair capabilities and derived the transform of the time to failure distribution.

Also in 1966, Liebowitz [26] and Downton [27] obtained some new results for parallel redundant systems. Considering the two-unit redundant system treated also by Belyayev and Gaver, Liebowitz plotted the ratio (called the improvement factor)

$$\frac{\text{MTSF}}{\lambda^{-1}}$$

for various repair disciplines, where λ^{-1} is the expected time to failure for each unit. He concluded that the difference between MTSF with repair to that without was independent of the repair distribution itself and depended only on the quantity E[R], where E[R] is the expected repair time for each component. Downton used the properties of semi-

Markov processes to analyze a parallel redundant system composed of n units. Assuming exponential failure and general repair, Downton obtained the Laplace transform of the distribution of $T_{k,n}$, the time to failure for a k-out-of-n system, and $E[T_{k,n}]$ for $n \geq k$, $k = 2,3,4$.

The role of priorities in a slightly more complex system than those described above was treated by Natarajan [28] in 1967. Two paralleled radars working in conjunction with two paralleled computers describe the system which is considered operative so long as at least one computer and one radar are working. Natarajan derived the distribution of the time to failure and the MTSF assuming failures and repairs were exponential. Values of the MTSF were compared for the case of no priorities and when the preemptive-resume discipline was assumed. In his doctoral dissertation [29] completed in 1968, Natarajan treated a large class of reliability models. Both single and multiple repair facilities were discussed for a system composed of a basic unit supported by n-1 standbys, where the standbys may deteriorate in storage (sometimes called warm standbys).

Also in 1968, Mine, Osaki and Asakura [30] derived the MTSF for a 2-out-of-n system by considering first passage times. The improvement factor was evaluated and its asymptotic behavior was studied.

In 1969, Epstein [31] considered a two-unit parallel redundant system where failure was exponential (but with a different failure rate for each unit) and repair was general. Transforms of distributions were obtained for the time to system failure, the total time spent on repair and the free time of the repairman.

In 1970, Rao and Natarajan [32] presented a survey paper on the reliability of systems with hot, warm and cold standbys for both parallel redundant systems and standby redundant systems. Rao and Natarajan remarked that the problem of finding the distribution of the time to system failure for a k-out-of-n system has not been solved when either the repair or the failure distribution is not exponential. Also in 1970, Osaki, Muth and Mazumdar analyzed some two-unit systems. Osaki [33] treated a two-unit standby redundant system and found the MTSF using a state transition diagram (or signal flow graph) approach. Osaki [34] applied the integral equation of renewal theory to four two-unit redundant systems (parallel redundancy, standby redundancy with and without priorities and standby redundancy with noninstantaneous switchover). The transform of the time to system failure was derived in each case. Muth [35] took advantage of the regenerative properties (discussed extensively by Smith [36]) of the two-unit parallel redundant system, and derived the time to failure distribution and the MTSF in an elegant manner. Mazumdar [37] considered a two-unit redundant system where the operating unit and the standby may have different failure rates. Mazumdar found the MTSF when the detection of failure for the standby and the operating unit may not be found instantaneously.

In 1971, Gaver and Luckew [38] treated what they termed an accumulation model, where a system experiences accumulating degrading, but not fatal, failures. Joint transforms were derived relating total time spent on repair and the number of repairs made. Later in 1971, Branson and Shah [39] considered a two-unit parallel redundant system where each unit had a different but constant failure rate, and repair distributions were different but general for each unit. Using properties of semi-Markov processes, Branson and Shah dervied the MTSF for the two-unit system and discussed some of the difficulties encountered when the semi-Markov model was applied to a three-unit system under similar assumptions.

Problem Statement

Although many authors discuss properties of T, the time to system failure, other measures of system reliability have been generally neglected in the literature. This dissertation will exhibit techniques for finding transforms of distributions for the time spent on repairs during T, the time that the repairman is idle during T and the number of breakdowns prior to system failure. In addition, assuming random (exponential) failure, general repair capabilities and one repairman, the supplementary variable technique of Cox [19], Belyayev [17] and Gaver [18] will be extended to allow analysis of the k-out-of-n system. A comparison will be made between the supplementary variable approach used in this research and the semi-Markov model employed by Downton [27] and Branson and Shah [39].

Research Objectives

Specifically, the objectives of this dissertation can be summarized as follows:

(1) For the k-out-of-n system, to find the transform of the distribution of T assuming that λ_m and $F_m(\cdot)$ are, respectively, the failure rate and general repair distribution for machine m, $1 \leq m \leq n$.

(2) For the n-out-of-n system with $\lambda_m = \lambda$ and $F_m(\cdot) = F(\cdot)$, $1 \leq m \leq n$, to find a general expression for the MTSF.

(3) For the 2-out-of-n system and the two-unit standby redundant system, to derive transforms of distributions for the time spent on repairs during T, the time that the repairman is idle during T and the number of repairs completed prior to system failure. Also, assuming exponential failure and Erlangian repair capabilities, to analyze the distribution of T for each system and apply the results to a variation of the accumulation model.

(4) To provide the reader with insight into solutions of problems which are as yet unsolved; namely, the analysis of the 2-out-of-2 system when failures and repairs follow general distributions and the application of stochastic integrals to redundant systems.

Research Profile

In Chapter 2, the transform of the time to failure distribution is obtained for some k-out-of-n systems and a comparison is made between the supplementary variable approach and the semi-Markov method.

Chapter 3 concerns the 2-out-of-n system and the two-unit standby redundant system. A conditional transform approach is applied and the distribution of the time to system failure for both systems is analyzed.

Chapter 4 deals with applications of the research.

Conclusions and areas for future research are discussed in Chapter 5.

CHAPTER 2

THE k-OUT-OF-n SYSTEM

Introduction

In this chapter, the supplementary variable technique is used to find the transform of the time to system failure for the k-out-of-n system. After introducing some preliminaries, the k-out-of-n system is treated first, when $\lambda_m = \lambda > 0$, $F_m(\cdot) = F(\cdot)$, $1 \leq m \leq n$, and second, when λ_m and $F_m(\cdot)$ are distinct for each m. A comparison between the supplementary variable method and the semi-Markov approach is also discussed.

Assumptions, Definitions and Notation

The following will be assumed:

(1) n units (machines) are in parallel redundancy and the time to failure of each machine is independently and exponentially distributed. The failure rate of machine m is $\lambda_m > 0$, $1 \leq m \leq n$.

(2) At $t = 0$, all units are operative.

(3) There is only one repairman (with unlimited service capabilities) and failed units queue up for service on a "first-come, first-served" basis. Machine m is serviced according to a general repair distribution, $F_m(\cdot)$, and repair times are independent random variables which are also independent of the failure times.

(4) Repaired units are put back into operation if k-1 or less units are in a failed state. When k units are simultaneously inoperable for the first time, the system fails.

The following notation will be used:

(a) $T_{k,n}$ - a continuous random variable which for any k-out-of-n system represents the length of time elapsed before k machines are simultaneously inoperable for the first time, assuming all n machines are running at $t = 0$; $0 < T_{k,n} < \infty$.

(b) R_{mj} - continuous random variables which represent the time needed to repair the j-th breakdown of machine m; $0 < R_{mj} < \infty$, $m=1,2,\ldots,n$; $j=1,2,\ldots$.

(c) y - a continuous random variable which at any moment t represents the time already spent on the repair of the machine undergoing service at that moment; $0 \leq y \leq t < \infty$.

(d) i - a discrete random variable which at any moment t represents the number of machines broken down at that moment; $i=0,1,\ldots,k \leq n$.

(e) E - the event "$T_{k,n} > t$."

(f) Let $F_m(\cdot)$ denote the common distribution function for the random variables R_{mj}; i.e., $F_m(r) = P[R_{mj} \leq r]$, $m=1,2,\ldots,n$; $j=1,2,\ldots$.

(g) Put $P_{k,n}(t) = P[E]$ and hence, $P_{k,n}(0) = 1$ by assumption (2).

(h) Let $W_{k,n}(s)$ denote the Laplace transform of $P_{k,n}(t)$; i.e., $W_{k,n}(s) = \int_0^\infty e^{-st} P_{k,n}(t)dt$.

(i) Let $h_{k,n}(t)$ and $h^*_{k,n}(s)$ represent the density of $T_{k,n}$ and its Laplace transform, respectively.

(j) Let $X(t)$ represent the state of the k-out-of-n system at any moment t; e.g.,

(i) $X(t) = (i,z)$, $1 \leq i \leq k \leq n$, $z \geq 0$, means that at moment t, there are i machines awaiting repair (including the machine presently being serviced) and an amount of time z, $0 \leq z \leq t$, has already been spent on the repair of the machine presently being serviced.

(ii) $X(t) = (0,0)$ means that at moment t, there are no machines broken down (i.e., no machines awaiting or being repaired) and, consequently, no time has been spent on the repair of a machine.

(k) Put $p_0(t) = P[X(t) = (0,0)]$; hence, $p_0(t = 0) = 1$ by assumption (2).

(l) Let $p_i(t,z)$ represent continuous and differentiable functions such that $p_i(t,z)dz = P[X(t) = (i,y)]$, $1 \leq i \leq k \leq n$, $z < y \leq z + dz$.

(m) Define

$$V_0(s) = \int_0^\infty e^{-st} p_0(t)dt$$

and

$$V_i(s) = \int_0^\infty e^{-st}\left(\int_0^t p_i(t,z)dz\right)dt, \quad 1 \leq i \leq k-1$$

(n) By definitions (k), (l) and (g)

$$P_{k,n}(t) = p_0(t) + \sum_{i=1}^{k-1} \int_0^t p_i(t,z)dz$$

and hence, using (m) and (h)

$$W_{k,n}(s) = \sum_{i=0}^{k-1} V_i(s), \quad 1 \leq k \leq n$$

The Case of $\lambda_m = \lambda$ and $F_m(\cdot) = F(\cdot)$, $1 \leq m \leq n$

Preliminaries

From the above assumptions and definitions, transition probabilities for $X(\cdot)$ during the time interval $(t, t + dt)$ depend only on the state of the system at moment t. In particular, when

$$\lambda_m = \lambda > 0 \qquad \text{and} \qquad F_m(\cdot) = F(\cdot),\ 1 \leq m \leq n \tag{2.1}$$

the transition probabilities can be described as follows:

(i) $P[X(t + dt) = (1,0) \mid X(t) = (0,0)] = n\lambda dt + o(dt)$

(ii) $P[X(t + dt) = (0,0) \mid X(t) = (0,0)] = 1 - n\lambda dt + o(dt)$

(iii) $P[X(t + dt) = (j + 1, z + dt) \mid X(t) = (j,z)]$

$$= (n - j)\lambda dt\{(1 - F(z + dt))/(1 - F(z))\} + o(dt),\ 1 \leq j \leq k\text{-}1$$

(iv) $P[X(t + dt) = (j,z + dt) \mid X(t) = (j,z)]$

$$= (1 - (n-j)\lambda dt)\{(1 - F(z + dt))/(1 - F(z))\} + o(dt),\ 1 \leq j \leq k\text{-}1$$

(v) $P[X(t + dt) = (j-1,0) \mid X(t) = (j,z)]$

$$= (1 - (n-j)\lambda dt)\{(F(z + dt) - F(z))/(1 - F(z))\} + o(dt),\ 1 \leq j \leq k\text{-}1$$

The 2-Out-of-n System

Consider the 2-out-of-n system. For n=2, a typical realization of the process X(t) in terms of the random variables y and i is presented in Figure 2.1.

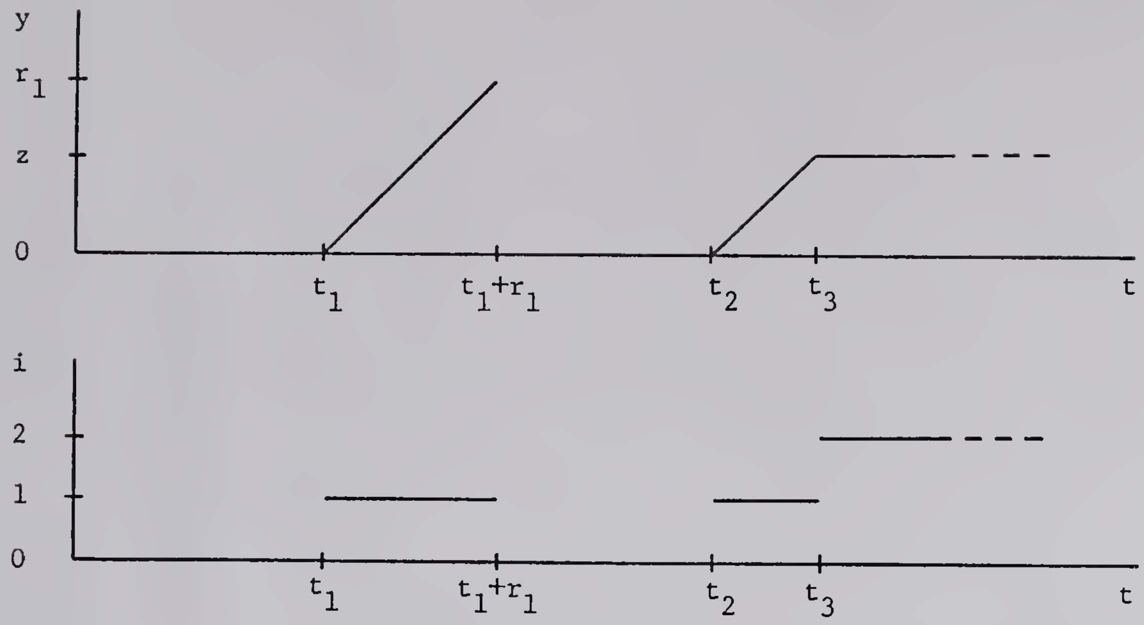

Figure 2.1. Sample function of the process $X(t) = (i,y)$, $i = 0, 1, 2$, $y \geq 0$. t_j is the moment at which the j-th breakdown occurs and r_1 is the time needed to complete the first repair. The second repair is not completed before the absorbing state (2,z) is reached, and thus, $T_{2,2} = t_3$.

Two quantities of interest are the distribution of the random variable $T_{2,n}$ and its expectation, $E[T_{2,n}]$. In this regard, from definitions (g) and (h) it is seen that

$$h_{2,n}(t) = \frac{d}{dt}(1 - P_{2,n}(t)) = -\frac{d}{dt} P_{2,n}(t) \tag{2.2}$$

and since $P_{2,n}(t=0) = 1$, from (2.2)

$$h^*_{2,n}(s) = 1 - sW_{2,n}(s) \tag{2.3}$$

Furthermore, assuming $E[T_{2,n}] < \infty$, from (2.3)

$$E[T_{2,n}] = (1 - h^*_{2,n}(s))/s\Big|_{s=0} = W_{2,n}(s)\Big|_{s=0} \tag{2.4}$$

To find $W_{2,n}(s)$, first consider the state transition equations. Using definitions (k) and (l), transition probabilities (i)-(v) and the fact that (2,z), $z \geq 0$, are absorbing states, the following relationships hold for $n \geq 2$:

$$p_0(t + dt) = p_0(t)(1 - n\lambda dt) + \\ + \int_0^t p_1(t,z)(1 - (n - 1)\lambda dt) \frac{F(z + dt) - F(z)}{1 - F(z)} dz + o(dt) \tag{2.5a}$$

$$p_1(t + dt, z + dt) = p_1(t,z)(1 - (n - 1)\lambda dt) \frac{1 - F(z + dt)}{1 - F(z)} + o(dt) \tag{2.5b}$$

$$p_1(t + dt, 0)dt = p_0(t)n\lambda dt + o(dt) \tag{2.5c}$$

$$p_0(0) = 1 \tag{2.5d}$$

Defining $g_1(t,z)$ by the relationship

$$p_1(t,z) = g_1(t,z)e^{-(n-1)\lambda z}(1 - F(z)) \tag{2.6}$$

and using the fact[1] that for $g(\cdot)$ continuous and $F(\cdot)$ absolutely continuous

$$\lim_{h \to 0} \int_0^\infty g(x) \frac{F(x + h) - F(x)}{h} dx = \int_0^\infty g(x)dF(x) \tag{2.7}$$

equations (2.5a) - (2.5d) may be rewritten as

---

[1] This can be shown by using the properties of continuous and absolutely continuous functions and appealing to the Lebesgue Dominated Convergence Theorem.

$$\frac{dp_0}{dt} = -n\lambda p_0(t) + \int_0^t e^{-(n-1)\lambda z} g_1(t,z)dF(z), p_0(0) = 1 \tag{2.8a}$$

$$\frac{\partial g_1}{\partial t} + \frac{\partial g_1}{\partial z} = 0 \tag{2.8b}$$

$$p_1(t,0) = n\lambda p_0(t) \tag{2.8c}$$

To obtain (2.8a), substitute (2.6) into (2.5a), use (2.7) and take the limit as dt → 0. To obtain (2.8b), first substitute (2.6) into (2.5b). Noting that $e^{-(n-1)\lambda dt} = 1 - (n - 1)\lambda dt + o(dt)$ obtain $g_1(t + dt,\ z + dt) = g_1(t,z) + o(dt)$. Adding in and subtracting out the quantities $g_1(t,z + dt)$, $g_1(t + dt,z)$ and $g_1(t,z)$ in the preceding equation, dividing by dt and taking the limit as dt → 0 yields (2.8b).

Equations (2.8a), (2.8b) and (2.8c) will now be used to find $W_{2,n}(s)$. Before doing so, note that by definition (n)

$$W_{2,n}(s) = V_0(s) + V_1(s), \quad n \geq 2 \tag{2.9}$$

Also, let the Laplace-Stieltjes transform of F(z) be $\phi(s) = \int_0^\infty e^{-sz}dF(z)$ and define

$$\psi_j(s) = \phi(s + j\lambda), \quad j \geq 1$$

Now, $W_{2,n}(s)$ can be found from equations (2.8a), (2.8b) and (2.8c) as follows. From (2.8b)

$$g_1(t,z) = g_1(t - z) \tag{2.10}$$

Combining (2.6) and (2.10) with (2.8c)

$$g_1(t) = n\lambda p_0(t) \tag{2.11}$$

Defining $G_1(s) = \int_0^\infty e^{-st} g_1(t)dt$, take the Laplace transform of (2.8a) and (2.11) and obtain respectively

$$V_0(s) = \frac{1 + \psi_{n-1}(s) \cdot G_1(s)}{s + n\lambda}$$

and

$$G_1(s) = n\lambda V_0(s)$$

or

$$V_0(s) = \{s + n\lambda(1 - \psi_{n-1}(s))\}^{-1} \tag{2.12}$$

From (2.6) and (2.10)

$$\begin{aligned} V_1(s) &= \int_0^\infty e^{-st}\left(\int_0^t e^{-(n-1)\lambda z} g_1(t - z)(1 - F(z))dz\right)dt \\ &= \frac{G_1(s)}{s + (n-1)\lambda}(1 - \psi_{n-1}(s)) \\ &= \frac{n\lambda(1 - \psi_{n-1}(s))}{(s + (n-1)\lambda)\{s + n\lambda(1 - \psi_{n-1}(s))\}} \end{aligned} \tag{2.13}$$

Thus, from (2.2), (2.3), (2.4), (2.12) and (2.13) it follows that for the 2-out-of-n system

$$W_{2,n}(s) = \{s + n\lambda(1 - \psi_{n-1}(s))\}^{-1} + \frac{n\lambda(1 - \psi_{n-1}(s))}{(s + (n-1)\lambda)\{s + n\lambda(1 - \psi_{n-1}(s))\}} \tag{2.14}$$

$$h^*_{2,n}(s) = 1 - sW_{2,n}(s)$$

and

$$E[T_{2,n}] = W_{2,n}(0) = \frac{1}{(n-1)\lambda} + \frac{1}{n\lambda(1 - \phi_{n-1})} \tag{2.15}$$

where $\phi_{n-1} = \phi((n-1)\lambda)$. Relation (2.15) agrees with equation (5.19a) in Downton [27].

The 3-Out-of-n System

For the 3-out-of-n system, $W_{3,n}(s)$ can be found in a similar manner. From definition (n)

$$W_{3,n}(s) = \sum_{i=0}^{2} V_i(s) \tag{2.16}$$

and as before

$$h^*_{3,n}(s) = 1 - sW_{3,n}(s)$$

and

$$E[T_{3,n}] = W_{3,n}(0)$$

Defining $g_1(t,z)$ and $q_2(t,z)$ by the relationships

$$p_1(t,z) = e^{-(n-1)\lambda z} g_1(t,z)(1 - F(z)) \tag{2.17}$$

$$q_2(t,z) = p_2(t,z)/(1 - F(z)) \tag{2.18}$$

the state transition equations for the 3-out-of-n system may be rewritten as

$$\frac{dp_0}{dt} = -n\lambda p_0(t) + \int_0^t e^{-(n-1)\lambda z} g_1(t,z)dF(z), p_0(0) = 1 \tag{2.19a}$$

$$\frac{\partial g_1}{\partial t} + \frac{\partial g_1}{\partial z} = 0 \tag{2.19b}$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial z} = -(n-2)\lambda q_2(t,z) + (n-1)\lambda e^{-(n-1)\lambda z} g_1(t,z) \tag{2.19c}$$

$$p_1(t,0) = n\lambda p_0(t) + \int_0^t q_2(t,z)dF(z) \tag{2.19d}$$

$$p_2(t,0) = 0 \tag{2.19e}$$

Again, equation (2.7) was used in obtaining equations (2.19a) and (2.19d). From (2.19a) - (2.19e), $W_{3,n}(s)$ can be obtained as follows. From (2.19b)

$$g_1(t,z) = g_1(t - z) \tag{2.20}$$

Substituting (2.20) into (2.19c) and solving yields

$$q_2(t,z) = e^{-(n-2)\lambda z} g_2(t - z) - (n-1)e^{-(n-1)\lambda z} g_1(t - z) \tag{2.21}$$

where $g_2(\cdot)$ is an arbitrary function of t - z. Equations (2.18) and (2.19e) imply that $q_2(t,0) = 0$, and thus, (2.21) becomes

$$q_2(t,z) = (n-1)g_1(t - z)\{e^{-(n-2)\lambda z} - e^{-(n-1)\lambda z}\} \tag{2.22}$$

Using (2.17) and (2.20)

$$p_1(t,0) = g_1(t,0) = g_1(t) \tag{2.23}$$

Combining (2.22) and (2.23) with (2.19d) yields

$$g_1(t) = n\lambda p_0(t) + (n-1)\int_0^t g_1(t - z)\{e^{-(n-2)\lambda z} - e^{-(n-1)\lambda z}\}dF(z) \tag{2.24}$$

$V_0(s)$ must be found from equations (2.19a) and (2.24). To this end, after taking the Laplace transform of (2.19a) and (2.24), obtain respectively

$$V_0(s) = \frac{1 + G_1(s) \cdot \psi_{n-1}(s)}{s + n\lambda} \tag{2.25}$$

and

$$G_1(s) = \frac{n\lambda V_0(s)}{1 - (n-1)\psi_{n-2}(s) + (n-1)\psi_{n-1}(s)} \tag{2.26}$$

where again

$$G_1(s) = \int_0^\infty e^{-st}\, g_1(t)\,dt$$

and

$$\psi_j(s) = \int_0^\infty e^{-(s+j\lambda)t}\,dF(t) = \phi(s + j\lambda), \quad j \geq 1$$

From (2.25) and (2.26)

$$V_0(s) = \frac{1 - (n-2)\psi_{n-2}(s) + (n-1)\psi_{n-1}(s)}{(s + n\lambda)\{1 - (n-1)\psi_{n-2}(s) + (n-1)\psi_{n-1}(s)\} - n\lambda\psi_{n-1}(s)} \tag{2.27}$$

Also, from (2.17) and the definition of $V_1(s)$

$$\begin{aligned} V_1(s) &= \int_0^\infty e^{-st}\left(\int_0^t e^{-(n-1)\lambda z}\, g_1(t - z)(1 - F(z))\,dz\right)dt \\ &= \frac{G_1(s)}{s + (n-1)\lambda}\,(1 - \psi_{n-1}(s)) \end{aligned} \tag{2.28}$$

Similarly, from (2.18), (2.22) and the definition of $V_2(s)$

$$\begin{aligned} V_2(s) &= (n-1)\int_0^\infty e^{-st}\left(\int_0^t g_1(t - z)\{e^{-(n-2)\lambda z} - e^{-(n-1)\lambda z}\}(1 - F(z))\,dz\right)dt \\ &= (n-1)G_1(s)\left\{ \frac{1 - \psi_{n-2}(s)}{s + (n-2)\lambda} - \frac{1 - \psi_{n-1}(s)}{s + (n-1)\lambda} \right\} \end{aligned} \tag{2.29}$$

Using (2.27), (2.28), (2.29) and (2.16), one finds after simplification

$$W_{3,n}(s) = [(s + n\lambda)(n-1)(n-2)\lambda \cdot$$

$$\cdot \{\lambda - (s + (n-1)\lambda)\psi_{n-2}(s) + (s + (n-2)\lambda)\psi_{n-1}(s)\}]/$$

$$/[(s + n\lambda)\{1 - (n-1)\psi_{n-2}(s) + (n-1)\psi_{n-1}(s)\} - n\lambda\psi_{n-1}(s)] \qquad (2.30)$$

and

$$h^*_{3,n}(s) = 1 - sW_{3,n}(s)$$

$$E[T_{3,n}] = W_{3,n}(0) = \frac{1}{(n-1)\lambda} + \frac{1}{(n-2)\lambda} + \frac{1 - \phi_{n-1} + \phi_{n-2}}{n\lambda\{1 - (n-1)\phi_{n-2} + (n-2)\phi_{n-1}\}} \qquad (2.31)$$

where $\phi_j = \psi_j(0) = \phi(j\lambda)$, $j \geq 1$. Equation (2.31) agrees with equation (5.19b) in Downton [27].

It should now be clear, however, that although $E[T_{k,n}]$ can be found for any fixed k, a general expression for $E[T_{k,n}]$ cannot be obtained by this approach; the reason of course being that the set of differential-integral equations which result from the state transition equations is different for each value of k. A similar difficulty encountered by Downton will be discussed below.

The n-Out-of-n System

Although unsuccessful in obtaining a general expression for $E[T_{k,n}]$ it will now be shown that the problem of finding $E[T_{n,n}]$ can be reduced to solving a system of $(n-1) \times (n-1)$ linear equations. First, for $n \geq 4$ (the n=2 and n=3 cases follow from equations (2.15) and (2.31) above), the state equations are

$$p_0(t + dt) = p_0(t)(1 - n\lambda dt) + \int_0^t p_1(t,z)(1 - (n-1)\lambda dt) \cdot$$

$$\cdot \frac{F(z + dt) - F(z)}{1 - F(z)} dz + o(dt) \tag{2.32a}$$

$$p_1(t + dt, z + dt) = p_1(t,z)(1 - (n-1)\lambda dt) \frac{1 - F(z + dt)}{1 - F(z)} + o(dt) \tag{2.32b}$$

$$p_k(t + dt, z + dt) = p_k(t,z)(1 - (n-k)\lambda dt) \frac{1 - F(z + dt)}{1 - F(z)} +$$

$$+ p_{k-1}(t,z)(n - k + 1)\lambda dt \frac{1 - F(z + dt)}{1 - F(z)} + o(dt), \ 2 \leq k \leq n-1 \tag{2.32c}$$

$$p_n(t + dt, z + dt) = p_n(t,z) + p_{n-1}(t,z)\lambda dt \frac{1 - F(z + dt)}{1 - F(z)} + o(dt) \tag{2.32d}$$

$$p_1(t + dt, 0)dt = p_0(t)n\lambda dt + \int_0^t p_2(t,z)(1 - (n-2)\lambda dt) \cdot$$

$$\cdot \frac{F(z + dt) - F(z)}{1 - F(z)} dz + o(dt) \tag{2.32e}$$

$$p_k(t + dt, 0)dt = \int_0^t p_{k+1}(t,z)(1 - (n-k-1)\lambda dt) \frac{F(z + dt) - F(z)}{1 - F(z)} dz +$$

$$+ o(dt), \quad 2 \leq k \leq n-2 \tag{2.32f}$$

$$p_j(t + dt, 0)dt = o(dt), \quad j=n-1,n \tag{2.32g}$$

$$p_0(0) = 1 \tag{2.32h}$$

Proceeding similarly as above, define $g_1(t,z)$ and $\{q_k(t,z)\}_{k=2}^{n-1}$ by the relationships

$$p_1(t,z) = g_1(t,z)e^{-(n-1)\lambda z}(1 - F(z)) \tag{2.33}$$

$$p_k(t,z) = q_k(t,z)(1 - F(z)), \quad 2 \leq k \leq n-1 \tag{2.34}$$

Now, using again the fact that for $g(\cdot)$ continuous and $F(\cdot)$ absolutely continuous

$$\lim_{h\to 0}\int_0^\infty g(x)\,\frac{F(x+h)-F(x)}{h}\,dx = \int_0^\infty g(x)\,dF(x)$$

equations (2.32a) - (2.32h) may be rewritten with the aid of (2.33) and (2.34) as

$$\frac{dp_0}{dt} = -n\lambda p_0(t) + \int_0^t e^{-(n-1)\lambda z}\, g_1(t,z)\,dF(z),\, p_0(0) = 1 \tag{2.35a}$$

$$\frac{\partial g_1}{\partial t} + \frac{\partial g_1}{\partial z} = 0 \tag{2.35b}$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial z} = -\,(n-2)\lambda q_2(t,z) + (n-1)\lambda e^{-(n-1)\lambda z}\, g_1(t,z) \tag{2.35c}$$

$$\frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial z} = -\,(n-k)\lambda q_k(t,z) + (n-k+1)\lambda q_{k-1}(t,z),\; 3 \le k \le n-1 \tag{2.35d}$$

$$\frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial z} = \lambda q_{n-1}(t,z)(1 - F(z))$$

$$p_1(t,0) = n\lambda p_0(t) + \int_0^t q_2(t,z)\,dF(z) \tag{2.35e}$$

$$p_k(t,0) = \int_0^t q_{k+1}(t,z)\,dF(z),\quad 2 \le k \le n-2 \tag{2.35f}$$

$$p_j(t,0) = 0,\quad j = n-1, n \tag{2.35g}$$

Equations (2.35a) - (2.35g) will now be solved and the results used to find $W_{n,n}(s)$. First, let the solution of

$$\frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial z} + \alpha q_k(t,z) = f(t,z),\quad 2 \le k \le n-1$$

be of the form

$$q_k(t,z) = e^{-\alpha z} g_k(t - z) + q_k^p(t,z)$$

where the $g_k(t - z)$ are functions whose existence and uniqueness follow from the existence and uniqueness of the $\{p_k(t,z)\}_{k=2}^{n-1}$ and equation (2.34), and $q_k^p(t,z)$ is the appropriate particular solution.

Theorem 2.1. The solution of (2.35c) and (2.35d) in terms of the $\{g_k(t - z)\}_{k=2}^{n-1}$ is

$$q_k(t,z) = \sum_{i=0}^{k-1} (-1)^i \binom{n-k+i}{i} e^{-(n-k+i)\lambda z} g_{k-i}(t - z) \qquad (2.36)$$

for $2 \leq k \leq n-1$, $n \geq 4$.

Proof. The proof consists of substituting (2.36) into (2.35c) and (2.35d) directly. ||

Now, from (2.35e) - (2.35g), (2.33) and (2.34), it follows that

$$g_1(t) = p_1(t,0) = n\lambda p_0(t) + \int_0^t q_2(t,z)dF(z) \qquad (2.37)$$

$$q_{k-1}(t,0) = p_{k-1}(t,0) = \int_0^t q_k(t,z)dF(z), \quad 3 \leq k \leq n-1 \qquad (2.38)$$

$$q_{n-1}(t,0) = p_{n-1}(t,0) = 0 \qquad (2.39)$$

Substituting (2.36) into (2.37), (2.38) and (2.39), and taking Laplace transforms of the resultant equations and (2.35a) yields a system of n-1 equations in terms of the n-1 unknowns, $V_0(s)$, $G_1(s)$, $G_2(s),\ldots,G_{n-2}(s)$, $n \geq 4$, where

$$G_j(s) = \int_0^\infty e^{-st} g_j(t)dt, \ 1 \le j \le n-2$$

Remembering that $\psi_j(s) = \int_0^\infty \exp\{-(s + j\lambda)t\}dF(t)$, the set of $n-1$ equations in $n-1$ unknowns can be written in matrix notation as

$$\underline{D_n(s)} \cdot \underline{g_n(s)} = \underline{e}_1 \tag{2.40}$$

where

$$\underline{g_n}(s)' = (V_0(s), G_1(s), G_2(s), \ldots, G_{n-2}(s))$$

$$\underline{e}_1' = (1,0,0,\ldots,0)$$

and, dropping the s's from the ψ_j's, reference the matrix on page 24.

Thus, for every fixed $n \ge 4$, equation (2.40) may be solved for $V_0(s), G_1(s), \ldots, G_{n-2}(s)$ in terms of the $\psi_j(s)$, $1 \le j \le n-1$. Using (2.36) and the definitions of $\{V_k(s)\}_{k=1}^{n-1}$

$$V_1(s) = \frac{G_1(s)}{s + (n-1)\lambda}(1 - \psi_{n-1}(s))$$

$$V_k(s) = \sum_{i=0}^{k-1} (-1)^i \binom{n-k+i}{i} \frac{G_{k-i}(s)(1 - \psi_{n-k+i}(s))}{s + (n-k+i)\lambda}, \ 2 \le k \le n-2$$

$$V_{n-1}(s) = \sum_{i=2}^{n-1} \left[i(-1)^i G_{n-i}(s) \left[\frac{1 - \psi_1(s)}{s + \lambda} - \frac{1 - \psi_i(s)}{s + i\lambda} \right] \right]$$

Now, by the definition of $W_{n,n}(s)$ and the above

$$W_{n,n}(s) = V_0(s) + V_1(s) + \sum_{k=2}^{n-2} V_k(s) + V_{n-1}(s)$$

$$= V_0(s) + \frac{G_1(s)(1 - \psi_{n-1}(s))}{s + (n-1)\lambda} +$$

$$\underline{D_n(s)} = \begin{pmatrix} s+n\lambda & -\psi_{n-1} & 0 & 0 & 0 & \cdots & 0 \\ n\lambda & -\left[\binom{n-1}{0}+\binom{n-1}{1}\psi_{n-1}\right] & \psi_{n-2} & 0 & 0 & \cdots & 0 \\ 0 & \binom{n-1}{1}+\binom{n-1}{2}\psi_{n-1} & -\left[\binom{n-2}{0}+\binom{n-2}{1}\psi_{n-2}\right] & \psi_{n-3} & 0 & \cdots & 0 \\ . & \cdots\cdots\cdots & \cdots\cdots & \cdots & . & \cdots & \cdots\cdots \\ 0 & (-1)^{n-1}\left[\binom{n-1}{n-4}+\binom{n-1}{n-3}\psi_{n-1}\right] & (-1)^{n-2}\left[\binom{n-2}{n-5}+\binom{n-2}{n-4}\psi_{n-2}\right] & \cdots & & -\left[\binom{3}{0}+\binom{3}{1}\psi_3\right] & \psi_2 \\ 0 & (-1)^{n-1}\left[\binom{n-1}{n-2}[\psi_1-\psi_{n-1}]-\binom{n-1}{n-3}\right] & (-1)^{n-2}\left[\binom{n-2}{n-3}[\psi_1-\psi_{n-2}]-\binom{n-2}{n-4}\right] & \cdots & & -\left[\binom{3}{2}[\psi_1-\psi_3]-\binom{3}{1}\right] & \binom{2}{1}[\psi_1-\psi_2]-\binom{2}{0} \end{pmatrix}$$

$$+ \sum_{k=2}^{n-2} \left[\sum_{i=0}^{k-1} \left[(-1)^i \binom{n-k+i}{i} \frac{G_{k-i}(s)(1 - \psi_{n-k+i}(s))}{s + (n-k+i)\lambda} \right] \right] +$$

$$+ \sum_{i=2}^{n-1} \left[i(-1)^i G_{n-i}(s) \left[\frac{1 - \psi_1(s)}{s + \lambda} - \frac{1 - \psi_i(s)}{s + i\lambda} \right] \right]$$

or, after simplification

$$W_{n,n}(s) = V_0(s) + \sum_{j=1}^{n-2} \left[(-1)^{n-j} G_j(s) \cdot \right.$$

$$\left. \cdot \left[\frac{(n-j)(1 - \psi_1(s))}{s + \lambda} - \frac{1 - \psi_{n-j}(s)}{s + (n-j)\lambda} \right] \right] \qquad (2.41)$$

where, for each value of $n \geq 4$, $V_0(s)$, $G_1(s), \ldots, G_{n-2}(s)$ are computed from (2.40). Hence, from (2.40) and (2.41) one may compute

$$h^*_{n,n}(s) = 1 - sW_{n,n}(s)$$

and

$$E[T_{n,n}] = W_{n,n}(0)$$

for the n-out-of-n system.

Taking $n = 4$, for example, (2.40) becomes

$$\begin{pmatrix} s + 4\lambda & -\psi_3(s) & 0 \\ 4\lambda & -(1+3\psi_3(s)) & \psi_2(s) \\ 0 & 3(1-\psi_1(s)+\psi_3(s)) & -(1-2\psi_1(s)+2\psi_2(s)) \end{pmatrix} \begin{pmatrix} V_0(s) \\ G_1(s) \\ G_2(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (2.42)$$

Solving (2.42) for $V_0(s)$, $G_1(s)$, $G_2(s)$ and evaluating them at $s = 0$ yields

$$V_0(0) = \frac{(1 + \phi_3)(1 - 2\phi_1 + 2\phi_2) - 3\phi_2(1 - \phi_1 + \phi_3)}{4\lambda\{(1 + 2\phi_3)(1 - 2\phi_1 + 2\phi_2) - 3\phi_2(1 - \phi_1 + \phi_3)\}} \tag{2.43}$$

$$G_1(0) = \frac{1 - 2\phi_1 + 2\phi_2}{(1 + 2\phi_3)(1 - 2\phi_1 + 2\phi_2) - 3\phi_2(1 - \phi_1 + \phi_3)} \tag{2.44}$$

$$G_2(0) = \frac{3(1 - \phi_1 + \phi_3)}{(1 + 2\phi_3)(1 - 2\phi_1 + 2\phi_2) - 3\phi_2(1 - \phi_1 + \phi_3)} \tag{2.45}$$

where $\phi_i = \psi_i(0) = \phi(i\lambda)$, $i \geq 1$. Evaluating $W_{4,4}(s)$ from (2.41) at $s = 0$ and substituting (2.43) - (2.45), by algebra

$$E[T_{4,4}] = W_{4,4}(0)$$

$$= \frac{25 - 32\phi_1 - 49\phi_2 + 59\phi_3 + 63\phi_1\phi_2 + 19\phi_2\phi_3 - 82\phi_1\phi_3}{12\lambda\{1 - 2\phi_1 - \phi_2 + 2\phi_3 - 4\phi_1\phi_3 + \phi_2\phi_3 + 3\phi_1\phi_2\}} \tag{2.46}$$

Equation (2.46) agrees with equation (5.19c) in Downton [27].

The Case of λ_m, $F_m(\cdot)$ Distinct, $1 \leq m \leq n$

Preliminaries

When the restriction imposed by equation (2.1) is relaxed and λ_m and $F_m(\cdot)$ are different for each machine, let $\tilde{T}_{k,n}$ be the time to system failure. It will be shown that the problem of finding

$$E[e^{-s\tilde{T}_{k,n}}] = \tilde{h}^*_{k,n}(s)$$

and the MTSF, denoted now by $E[\tilde{T}_{k,n}]$, remains solvable for fixed values of k, $k \leq n$. The supplementary variable technique is still applicable but a redefinition of states is necessary.

In order to avoid new, or at best cumbersome, notation, the subscript i and the functions $p_0(t)$, $p_i(t,z)$, $g_i(t,z)$, $V_i(s)$, $G_i(s)$, $\psi_i(s)$ and ϕ_i will continue to be used, even though they may carry slightly different meanings than in previous sections. The reader is cautioned to make note of this whenever these quantities are defined below.

The 2-Out-of-n System

For this more general 2-out-of-n system, let $\tilde{X}(t)$ be the state of the system at any moment t; e.g.,

(i) $\tilde{X}(t) = (i,z)$, $1 \leq i \leq n$, $z \geq 0$, means that at moment t, machine number i is down, and an amount of time z, $0 \leq z \leq t$, has already been spent on the repair of machine i.

(ii) $\tilde{X}(t) = (0,0)$ means, as before, no machines are down.

Letting

$$p_0(t) = P[\tilde{X}(t) = (0,0)]$$

$$p_i(t,z)dz = P[\tilde{X}(t) = (i,y)],\ 1 \leq i \leq n,\ z < y \leq z + dz$$

$$\alpha = \sum_{j=1}^{n} \lambda_j,\quad \lambda^{(i)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \lambda_j,\quad 1 \leq i \leq n$$

the state transition equations can be written as follows

$$p_0(t + dt) = p_0(t)(1 - \alpha dt) + $$

$$+ \sum_{i=1}^{n} \int_0^t p_i(t,z)(1 - \lambda^{(i)}dt) \frac{F_i(z + dt) - F_i(z)}{1 - F_i(z)} dz + o(dt) \quad (2.47a)$$

$$p_i(t + dt, z + dt) = p_i(t,z)(1 - \lambda^{(i)}dt)\,\frac{1 - F_i(z + dt)}{1 - F_i(z)} + o(dt),\ 1 \le i \le n \tag{2.47b}$$

$$p_i(t + dt, 0)dt = p_0(t)\lambda_i dt + o(dt), \quad 1 \le i \le n \tag{2.47c}$$

$$p_0(0) = 1 \tag{2.47d}$$

Defining $\{g_i(t,z)\}_{i=1}^{n}$ by the relationships

$$p_i(t,z) = e^{-\lambda^{(i)}z}\, g_i(t,z)(1 - F_i(z)),\ 1 \le i \le n \tag{2.48}$$

equations (2.47a) - (2.47d) can be rewritten with the aid of relation (2.7) as

$$\frac{dp_0}{dt} = -\alpha p_0(t) + \sum_{i=1}^{n} \int_0^t e^{-\lambda^{(i)}z}\, g_i(t,z)dF_i(z), p_0(0) = 1 \tag{2.49a}$$

$$\frac{\partial g_i}{\partial t} + \frac{\partial g_i}{\partial z} = 0, \quad 1 \le i \le n \tag{2.49b}$$

$$g_i(t,0) = p_i(t,0) = \lambda_i p_0(t), \quad 1 \le i \le n \tag{2.49c}$$

For $1 \le i \le n$, put

$$V_0(s) = \int_0^\infty e^{-st} p_0(t)dt$$

$$V_i(s) = \int_0^\infty e^{-st}\left(\int_0^t p_i(t,z)dz\right)dt$$

$$G_i(s) = \int_0^\infty e^{-st} g_i(t)dt$$

$$\phi_i(s) = \int_0^\infty e^{-st} dF_i(t)$$

$$\psi_i(s) = \phi_i(s + \lambda^{(i)})$$

and

$$\tilde{W}_{2,n}(s) = \sum_{i=0}^{n} V_i(s)$$

Equations (2.49a) - (2.49c) can be used to find $\tilde{W}_{2,n}(s)$, and hence $E[\tilde{T}_{2,n}] = \tilde{W}_{2,n}(0)$, as follows. From (2.49b)

$$g_i(t,z) = g_i(t - z),\ 1 \leq i \leq n \tag{2.50}$$

Combining (2.50) with (2.49c) and taking Laplace transforms yields

$$G_i(s) = \lambda_i V_0(s),\ 1 \leq i \leq n \tag{2.51}$$

From (2.50), (2.51) and (2.49a), it follows that

$$V_0(s) = \{s + \alpha - \sum_{i=1}^{n} \lambda_i \psi_i(s)\}^{-1} \tag{2.52}$$

Using the definitions of $V_i(s)$, (2.48), (2.51) and (2.52)

$$\begin{aligned} V_i(s) &= \int_0^\infty e^{-st}\left(\int_0^t p_i(t,z)dz\right)dt \\ &= \frac{\lambda_i V_0(s)}{s + \lambda^{(i)}}(1 - \psi_i(s)) \\ &= \frac{\lambda_i(1 - \psi_i(s))}{(s + \lambda^{(i)})\{s + \alpha - \sum_{i=1}^{n} \lambda_i \psi_i(s)\}},\ 1 \leq i \leq n \end{aligned} \tag{2.53}$$

By (2.52) and (2.53)

$$\begin{aligned} \tilde{W}_{2,n}(s) &= \sum_{i=0}^{n} V_i(s) \\ &= \{s + \alpha - \sum_{i=1}^{n} \lambda_i \psi_i(s)\}^{-1} \left\{1 + \sum_{i=1}^{n} \frac{\lambda_i(1 - \psi_i(s))}{s + \lambda^{(i)}}\right\} \end{aligned} \tag{2.54}$$

and hence the transform of the density of $\tilde{T}_{2,n}$, $\tilde{h}^*_{2,n}(s)$, and $E[\tilde{T}_{2,n}]$ can be computed from (2.54) as

$$\tilde{h}^*_{2,n}(s) = 1 - s\tilde{W}_{2,n}(s) \tag{2.55}$$

and

$$\begin{aligned} E[\tilde{T}_{2,n}] &= \tilde{W}_{2,n}(0) \\ &= \{\alpha - \sum_{i=1}^{n} \lambda_i \phi_i\}^{-1} \left(1 + \sum_{i=1}^{n} \frac{\lambda_i(1 - \phi_i)}{\lambda^{(i)}}\right) \end{aligned} \tag{2.56}$$

where $\phi_i = \phi_i(\lambda^{(i)})$, $i \geq 1$. As a check, when $\lambda_i = \lambda$ and $F_i(\cdot) = F(\cdot)$, $\phi_i(s) = \phi(s)$, $\lambda^{(i)} = (n-1)\lambda$, $\psi_i(s) = \phi(s + (n-1)\lambda)$, $\alpha = n\lambda$ and (2.54) becomes

$$\begin{aligned} \tilde{W}_{2,n}(s) &= \{s + n\lambda[1 - \phi(s + (n-1)\lambda)]\}^{-1} \left(1 + \frac{n\,[1 - \phi(s + (n-1)\lambda)]}{s + (n-1)\lambda}\right) \\ &= \frac{1}{s + n\lambda[1 - \phi(s + (n-1)\lambda)]} + \frac{n\lambda[1 - \phi(s + (n-1)\lambda)]}{(s + (n-1)\lambda)\{s + n\lambda[1 - \phi(s+(n-1)\lambda)]\}} \\ &= W_{2,n}(s) \end{aligned}$$

as in equation (2.14) above. For n=2, equations (2.55) and (2.56) agree with equations (8) and (9) in Osaki [34]. Using the principle of regeneration, an alternative derivation of (2.55) is treated in the Appendix.

The 3-Out-of-n System

For the 3-out-of-n system, finding the expectation of $\tilde{T}_{3,n}$ again necessitates an expanded state space. As before, let $\tilde{X}(t)$ be the state of the system at any moment t; e.g.,

(i) $\tilde{X}(t) = (i,z)$, $1 \le i \le n$, $z \ge 0$, means that at moment t, machine number i is down and an amount of time z, $0 \le z \le t$, has already been spent on the repair of machine i.

(ii) $\tilde{X}(t) = (i,j;z)$, $1 \le i, j \le n$, $i \ne j$, $0 \le z \le t$, means that at moment t, machine number i and machine number j are down, and an amount of time z, $0 \le z \le t$, has already been spent on the repair of unit i.

(iii) Again, $\tilde{X}(t) = (0,0)$ means that at moment t, no machines are down.

For $1 \le i, j \le n$, $i \ne j$, let

$$p_0(t) = P[\tilde{X}(t) = (0,0)]$$

$$p_i(t,z)dz = P[\tilde{X}(t) = (i,y)],\ z < y \le z + dz$$

$$p_{ij}(t,z)dz = P[\tilde{X}(t) = (i,j;y)],\ z < y \le z + dz$$

$$\alpha = \sum_{j=1}^{n} \lambda_j, \quad \lambda^{(i)} = \sum_{\substack{j=1\\ j\ne i}}^{n} \lambda_j, \quad \lambda^{(ij)} = \sum_{\substack{k=1\\ i\ne j\ne k}}^{n} \lambda_k$$

With $S = S(i,j) = \{i,j: i,j = 1,2,\ldots,n;\ i \ne j\}$, the state transition equations are

$$p_0(t + dt) = p_0(t)(1 - \alpha dt) + \sum_{i=1}^{n} \int_0^t p_i(t,z)(1 - \lambda^{(i)}dt)\, \frac{F_i(z + dt) - F_i(z)}{1 - F_i(z)}\, dz + o(dt) \quad (2.57a)$$

$$p_i(t + dt, z + dt) = p_i(t,z)(1 - \lambda^{(i)}dt)\, \frac{1 - F_i(z + dt)}{1 - F_i(z)} + o(dt),\ 1 \le i \le n \quad (2.57b)$$

$$p_{ij}(t + dt, z + dt) = p_{ij}(t,z)(1 - \lambda^{(ij)}dt)\,\frac{1 - F_i(z + dt)}{1 - F_i(z)} +$$

$$+ p_i(t,z)\lambda_j dt\,\frac{1 - F_i(z + dt)}{1 - F_i(z)} + o(dt),\ i,j \in S \tag{2.57c}$$

$$p_i(t + dt, 0)dt = p_0(t)\lambda_i dt +$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} \int_0^t p_{ji}(t,z)(1 - \lambda^{(ij)}dt)\,\frac{F_j(z + dt) - F_j(z)}{1 - F_j(z)} +$$

$$+ o(dt),\ 1 \le i \le n \tag{2.57d}$$

$$p_{ij}(t + dt, 0)dt = o(dt),\ i,j \in S \tag{2.57e}$$

$$p_0(0) = 1 \tag{2.57f}$$

Defining the functions $\{g_i(t,z)\}_{i=1}^{n}$ and $q_{ij}(t,z)$, $i,j \in S$, by the relationships

$$p_i(t,z) = e^{-\lambda^{(i)}z} g_i(t,z)(1 - F_i(z)),\ 1 \le i \le n \tag{2.58}$$

$$p_{ij}(t,z) = q_{ij}(t,z)(1 - F_i(z)),\ i,j \in S \tag{2.59}$$

equations (2.57a) - (2.57f) may be rewritten using (2.7) as

$$\frac{dp_0}{dt} = -\alpha p_0(t) + \sum_{i=1}^{n} \int_0^t e^{-\lambda^{(i)}z} g_i(t,z)dF_i(z), p_0(0) = 1 \tag{2.60a}$$

$$\frac{\partial g_i}{\partial t} + \frac{\partial g_i}{\partial z} = 0,\quad 1 \le i \le n \tag{2.60b}$$

$$\frac{\partial q_{ij}}{\partial t} + \frac{\partial q_{ij}}{\partial z} = -\lambda^{(ij)} q_{ij}(t,z) + \lambda_j e^{-\lambda^{(i)}z} g_i(t,z),\ i,j \in S \tag{2.60c}$$

$$g_i(t,0) = p_i(t,0)$$

$$= \lambda_i p_0(t) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_0^t q_{ji}(t,z)dF_j(z), \; 1 \leq i \leq n \qquad (2.60d)$$

$$q_{ij}(t,0) = p_{ij}(t,0) = 0, \; i,j \in S \qquad (2.60e)$$

Now, for i,j ε S, put

$$V_0(s) = \int_0^\infty e^{-st} p_0(t)dt$$

$$V_i(s) = \int_0^\infty e^{-st}\left(\int_0^t p_i(t,z)dz\right)dt$$

$$V_{ij}(s) = \int_0^\infty e^{-st}\left(\int_0^t p_{ij}(t,z)dz\right)dt$$

$$G_i(s) = \int_0^\infty e^{-st} g_i(t)dt$$

$$\phi_i(s) = \int_0^\infty e^{-st} dF_i(t)$$

$$\psi_i(s) = \phi_i(s + \lambda^{(i)})$$

$$\psi_{ij}(s) = \phi_i(s + \lambda^{(ij)})$$

and

$$\tilde{W}_{3,n}(s) = \sum_{i=0}^{n} V_i(s) + \sum_{i,j} \sum_{\varepsilon\, S} V_{ij}(s)$$

Equations (2.60a) - (2.60e) will now be used to find $\tilde{W}_{3,n}(s)$ and $\tilde{h}^*_{3,n}(s) = E[e^{-s\tilde{T}_{3,n}}]$, as follows. From (2.60b)

$$g_i(t,z) = g_i(t - z), \; 1 \leq i \leq n \qquad (2.61)$$

Using (2.61) and (2.60e) the solution of (2.60c) becomes

$$q_{ij}(t,z) = e^{-\lambda^{(ij)}z}g_i(t - z) - e^{-\lambda^{(i)}z}g_i(t - z), \; i,j \in S \qquad (2.62)$$

Substituting (2.61) and (2.62) into (2.60d) and taking Laplace transforms of the resultant equations, obtain

$$G_i(s) = \lambda_i V_0(s) + \sum_{\substack{j=1 \\ j\neq i}}^{n} (\psi_{ji}(s) - \psi_j(s)) \cdot G_j(s), \; 1 \leq i \leq n \qquad (2.63)$$

Transforming both sides of (2.60a) with the aid of (2.61) yields

$$(s + \alpha)V_0(s) = 1 + \sum_{i=1}^{n} \psi_i(s) \cdot G_i(s) \qquad (2.64)$$

Equations (2.63) and (2.64) represent n+1 relationships between the n+1 functions $V_0(s)$, $G_1(s)$, $G_2(s),\ldots,G_n(s)$.

From (2.58) and the definitions of the $V_i(s)$

$$V_i(s) = \frac{G_i(s)}{s + \lambda^{(i)}} (1 - \psi_i(s)), \; 1 \leq i \leq n \qquad (2.65)$$

and from (2.59), (2.62) and the definitions of the $V_{ij}(s)$

$$V_{ij}(s) = \frac{G_i(s)}{s + \lambda^{(ij)}} (1 - \psi_{ij}(s)) - \frac{G_i(s)}{s + \lambda^{(i)}} (1 - \psi_i(s)), \; i,j \in S \qquad (2.66)$$

Hence, using (2.65) and (2.66), for $n \geq 3$

$$\begin{aligned}\tilde{W}_{3,n}(s) &= V_0(s) + \sum_{i=1}^{n} V_i(s) + \sum_{i,j}\sum_{\in S} V_{ij}(s) \\ &= V_0(s) + \sum_{i,j}\sum_{\in S} \left(\frac{G_i(s)(1 - \psi_{ij}(s))}{s + \lambda^{(ij)}} \right) - (n-2) \sum_{i=1}^{n} \frac{G_i(s)}{s+\lambda^{(i)}} (1-\psi_i(s))\end{aligned} \qquad (2.67)$$

$$\tilde{h}^*_{3,n}(s) = 1 - s\tilde{W}_{3,n}(s) \qquad (2.68)$$

and

$$E[\tilde{T}_{3,n}] = \tilde{W}_{3,n}(0) \qquad (2.69)$$

where $V_0(s)$ and $\{G_i(s)\}_{i=1}^n$ must be found from equations (2.63) and (2.64).

As a check on (2.67), let $\lambda_i = \lambda$ and $F_i(\cdot) = F(\cdot)$, $1 \leq i \leq n$.

Then

$$\phi_i(s) = \phi(s),\ \psi_i(s) = \phi(s + (n-1)\lambda),\ \psi_{ij}(s) = \phi(s + (n-2)\lambda),$$

$$\alpha = n\lambda,\ \lambda^{(i)} = (n-1)\lambda,\ \lambda^{(ij)} = (n-2)\lambda,\ i,j \in S$$

For convenience, let

$$\beta_{n-1}(s) = \phi(s + (n-1)\lambda)$$

$$\beta_{n-2}(s) = \phi(s + (n-2)\lambda)$$

$$\gamma_n(s) = \beta_{n-2}(s) - \beta_{n-1}(s)$$

In terms of the above notation, equations (2.63) and (2.64) can be written in matrix notation as (equation, page 36). Using induction and Cramer's rule on the above, for $n \geq 3$

$$V_0(s) = \frac{1 - (n-1)\gamma_n(s)}{(s + n\lambda)(1 - (n-1)\gamma_n(s)) - n\lambda\beta_{n-1}(s)} \qquad (2.70)$$

and for $1 \leq i \leq n$

$$G_i(s) = \frac{\lambda}{(s + n\lambda)(1 - (n-1)\gamma_n(s)) - n\lambda\beta_{n-1}(s)} \qquad (2.71)$$

After substituting (2.70) and (2.71) into (2.67) and simplifying, it is seen that

$$\begin{pmatrix} s + n\lambda & -\beta_{n-1}(s) & -\beta_{n-1}(s) & -\beta_{n-1}(s) & \cdots\cdot & -\beta_{n-1}(s) \\ \lambda & -1 & \gamma_n(s) & \gamma_n(s) & \cdots\cdot & \gamma_n(s) \\ \lambda & \gamma_n(s) & -1 & \gamma_n(s) & \cdots\cdot & \gamma_n(s) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \gamma_n(s) & \gamma_n(s) & \gamma_n(s) & \cdots\cdot & -1 \end{pmatrix} \begin{pmatrix} V_0(s) \\ G_1(s) \\ G_2(s) \\ \vdots \\ G_n(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\tilde{W}_{3,n}(s) = W_{3,n}(s)$$

as in equation (2.30). Thus, for the special case of $\lambda_i = \lambda$ and $F_i(\cdot) = F(\cdot)$, equation (2.69) agrees with equation (5.19b) of Downton [27].

Again, however, since the state equations change for each value of k, a general expression for

$$\tilde{W}_{k,n}(s)$$

cannot be found with this technique. Nevertheless, for any fixed value of k, $1 \leq k \leq n$, an expression is obtainable for $\tilde{W}_{k,n}(s)$ and, hence, for

$$E[e^{-s\tilde{T}_{k,n}}] = 1 - s\tilde{W}_{k,n}(s)$$

and

$$E[\tilde{T}_{k,n}] = \tilde{W}_{k,n}(0)$$

A Comparison Between the Semi-Markov Model and the Supplementary Variable Technique

General

Since both Downton [27] and Branson and Shah [39] use semi-Markov processes to model repairable systems with standbys, a comparison is in order between the semi-Markov approach and the supplementary variable technique used in this research.

The Approach of Branson and Shah

In the two-unit system considered by Branson and Shah, a unit may operate on-line or off-line. The on-line unit, no matter which unit it happens to be, always fails with rate λ_1 and is repaired according to distribution function $F_1(\cdot)$. The same is true of the off-line unit with respect to λ_2 and $F_2(\cdot)$. This is not quite the same as the 2-out-of-2 system, but it is similar.

By a judicious choice of states, Branson and Shah modeled their two-unit system as a semi-Markov process. Then, using a result (Barlow and Proschan [9], Theorem 2.5, p. 135) which does not depend on the distribution of the time to system failure itself, the MTSF was found, given that the process began in any of the possible up-states.

The Semi-Markov Model of Downton

Downton used a semi-Markov process to model the k-out-of-n system where λ and $F(\cdot)$ are, respectively, the failure rate and repair distribution for each of the n units.

For $n \geq k$ and $k = 2,3$, Downton derived

$$E[e^{-sT_{k,n}} | 0 \text{ units are down initially}] \tag{2.72}$$

but these results are special cases of equations (2.55) and (2.68) with $\lambda_i = \lambda$ and $F_i(\cdot) = F(\cdot)$. In fairness to Downton, however, although (2.72) is often the case of interest, the semi-Markov approach also yields

$$E[e^{-sT_{k,n}} | i \text{ units are initially down, } 0 < i < k] \tag{2.73}$$

which cannot be obtained using the supplementary variable approach above.

For each value of k, the approach used by Downton necessitated a matrix inversion to find the transform of the appropriate first passage time distribution. Since the matrix to be inverted was different for each k, a general expression for $E[T_{k,n}]$ was not obtained. In the supplementary variable approach, it was the change in the state transition equations for each k which prevented a generalization of $E[T_{k,n}]$.

Summarizing Remarks.

Because the semi-Markov process is defined on a denumerable number of states, transforms of the distribution of T, the time to system failure, can be found when the process is in any of the up-states initially as indicated in (2.73). Although this is not the case with the supplementary variable approach, results implied by (2.72) can be obtained for a more general class of k-out-of-n systems than those treated by Downton.

Since Branson and Shah's approach to finding the MTSF is independent of the distribution of T itself, other moments of the time to system failure are not readily available. In this sense, the technique employed by Branson and Shah is less general than that of Downton or the author.

CHAPTER 3

SOME RELIABILITY CHARACTERISTICS OF THE 2-OUT-OF-n SYSTEM AND THE TWO-UNIT STANDBY REDUNDANT SYSTEM

Introduction

Reliability attributes of the 2-out-of-n and two-unit standby redundant systems are treated in this chapter. Besides the distribution of the number of repairs completed prior to system failure, transforms of distributions are derived for the time the repairman is idle during T (the time to system failure) and the time spent on repair during T. Assuming random failure and k-Erlang repair capabilities, the distribution of T for each system is analyzed and the generating function of the distribution of the number of renewals occurring during T is obtained.

The 2-Out-of-n System

Assumptions and Definitions

The following will be assumed:

(1) The time to failure of each unit is independently and exponentially distributed with failure rate $\lambda > 0$.

(2) At $t=0$, all n units are operating properly.

(3) There is only one repairman (with unlimited service capabilities) and the repair times are independent random variables which are also independent of the failure times.

Define:

$Z(t)$ - the number of units down at moment t ($Z(t) = 0$, 1 or 2).

$T_{2,n}$ - the time to system failure for the 2-out-of-n system. As before, the density of $T_{2,n}$ and its Laplace transform will be denoted by $h_{2,n}(t)$ and $h^*_{2,n}(s)$, respectively.

X_i - the time elapsed between the completion of the (i-1)-st repair and the next breakdown, $i \geq 1$ (the zeroth repair is assumed to be completed at t=0). By assumptions (1) and (2), the $\{X_i\}$, $i \geq 1$, are independent and identically distributed (i.i.d.) as a random variable X with density function

$$f_X(x) = \lambda_0 e^{-\lambda_0 x} \quad , \quad x \geq 0$$

where $\lambda_0 = n\lambda$, n=2,3,....

R_i - the time required to perform the i-th repair, $i \geq 1$. By assumption (3), the $\{R_i\}$, $i \geq 1$, are i.i.d. as a random variable with distribution function $F(\cdot)$. Let $\phi(s)$ be the Laplace-Stieltjes transform of $F(\cdot)$; i.e.,

$$\phi(s) = \int_0^\infty e^{-st} dF(t)$$

Y - the time to the next failure when only n-1 units are operative. By assumptions (1) and (3), the density function of Y is

$$f_Y(y) = \lambda_1 e^{-\lambda_1 y} \quad , \quad y \geq 0$$

where $\lambda_1 = (n-1)\lambda$, n=2,3,....

N - the number of repairs completed prior to system failure.

R_c - the total time spent on completed repairs during $T_{2,n}$.

R - the total time spent on all repairs during $T_{2,n}$.

I - the total time that the repairman is idle during $T_{2,n}$.

A Conditional Transform Approach

Before discussing the distributions of N, R and I, it will be instructive to consider a typical realization of the process Z(t) (Figure 3.1).

Note that

P[a repair interval contains no failures]

$$= P[R < Y] = \int_0^\infty e^{-\lambda_1 t} dF(t) = \phi(\lambda_1)$$

Clearly, the distribution of N is geometric and

$$P[N = i] = \{\phi(\lambda_1)\}^i (1 - \phi(\lambda_1)), \; i=0,1,\ldots \tag{3.1}$$

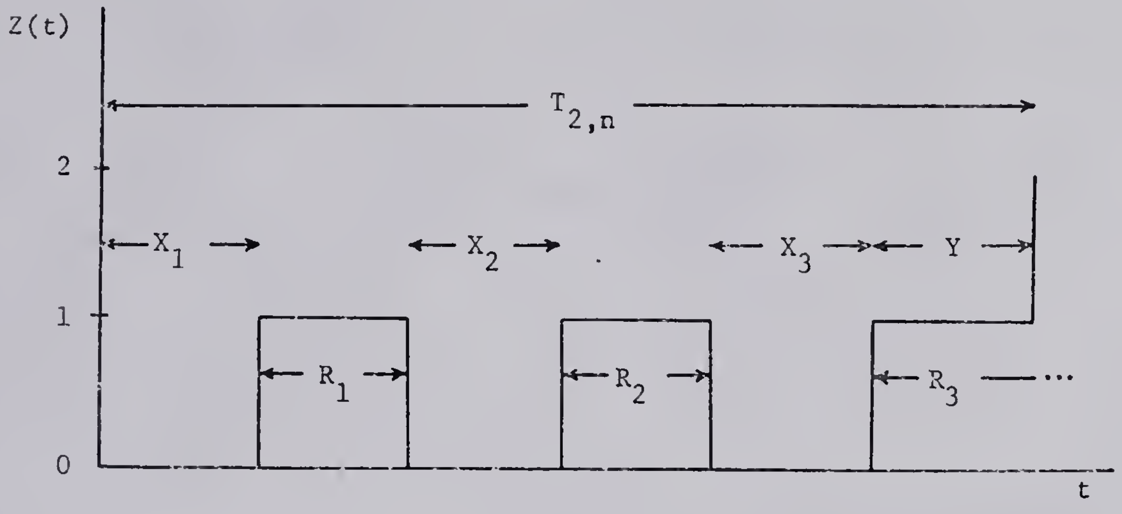

Figure 3.1. A sample function of Z(t) in terms of the random variables X_i, R_i, Y and $T_{2,n}$.

Given that N = i

$$R_c = \begin{cases} \tilde{R}_1 + \tilde{R}_2 + \ldots + \tilde{R}_i, & i=1,2,\ldots \\ 0, & i=0 \end{cases} \tag{3.2}$$

where the $\{\tilde{R}_j\}$, $j \geq 1$, are i.i.d. as a random variable $\tilde{R}$ with

$$dP[\tilde{R} \leq t] = \frac{e^{-\lambda_1 t} dF(t)}{\phi(\lambda_1)} \tag{3.3}$$

and

$$I = X_1 + X_2 + \ldots + X_{i+1}, \quad i \geq 0 \tag{3.4}$$

Hence, from (3.1), (3.2) and (3.3)

$$\begin{aligned} E[e^{-sR_c}] &= \sum_{i=0}^{\infty} E[e^{-sR_c} | N = i]P[N = i] \\ &= \sum_{i=0}^{\infty} \left(\frac{\phi(s + \lambda_1)}{\phi(\lambda_1)} \right)^i \{\phi(\lambda_1)\}^i (1 - \phi(\lambda_1)) \\ &= \frac{1 - \phi(\lambda_1)}{1 - \phi(s + \lambda_1)} \end{aligned} \tag{3.5}$$

and by (3.4) and the definition of the $\{X_j\}$, $j \geq 1$

$$\begin{aligned} E[e^{-sI}] &= \sum_{i=0}^{\infty} E[e^{-sI} | N = i]P[N = i] \\ &= \sum_{i=0}^{\infty} \left(\frac{\lambda_0}{s + \lambda_0} \right)^{i+1} \{\phi(\lambda_1)\}^i (1 - \phi(\lambda_1)) \\ &= \frac{\lambda_0(1 - \phi(\lambda_1))}{s + \lambda_0(1 - \phi(\lambda_1))} \end{aligned} \tag{3.6}$$

As a check, note that (see Figure 3.1)

$$T_{2,n} = Y^* + R_c + I \tag{3.7}$$

where

$$dP[Y^* \leq y] = \frac{\lambda_1 e^{-\lambda_1 y}(1 - F(y))}{1 - \phi(\lambda_1)}\, dy, \ y \geq 0$$

Now, by (3.2), (3.3) and (3.4)

$$E[e^{-s_1 R_c} e^{-s_2 I}]$$

$$= \sum_{i=0}^{\infty} E[e^{-s_1 R_c} e^{-s_2 I} | N = i] \cdot P[N = i]$$

$$= \sum_{i=0}^{\infty} \left(\frac{\phi(s_1 + \lambda_1)}{\phi(\lambda_1)} \right)^i \left(\frac{\lambda_0}{s_2 + \lambda_0} \right)^{i+1} (\phi(\lambda_1))^i (1 - \phi(\lambda_1))$$

$$= \frac{\lambda_0(1 - \phi(\lambda_1))}{s_2 + \lambda_0(1 - \phi(s_1 + \lambda_1))} \qquad (3.8)$$

Putting $s_1 = s_2 = s$ in (3.8)

$$E[e^{-s(R_c+I)}] = \frac{\lambda_0(1 - \phi(\lambda_1))}{s + \lambda_0(1 - \phi(s + \lambda_1))} \qquad (3.9)$$

By definition of $dP[Y^* \leq t]$ above

$$E[e^{-sY^*}] = \int_0^{\infty} e^{-st} dP[Y^* \leq t] = \frac{\lambda_1(1 - \phi(s + \lambda_1))}{(s + \lambda_1)(1 - \phi(\lambda_1))} \qquad (3.10)$$

Combining (3.9) and (3.10) with (3.7)

$$E[s^{-sT_{2,n}}] = E[e^{-sY^*}] \cdot E[e^{-s(R_c+I)}]$$

$$= \frac{\lambda_0 \lambda_1 [1 - \phi(s + \lambda_1)]}{(s + \lambda_1)(s + \lambda_0[1 - \phi(s + \lambda_1)])} \qquad (3.11)$$

and equation (3.11) agrees with (2.14).

Clearly, the moments

$$E[\{R_c\}^j\{I\}^k], \quad j,k=0,1,2,\ldots,$$

are available by differentiating (3.8). Also, since R, the total time spent on all repairs before failure, satisfies

$$R = R_c + Y^*$$

by (3.5) and (3.10)

$$E[e^{-sR}] = \frac{\lambda_1}{s + \lambda_1} \tag{3.12}$$

From (3.12) it is seen that R is distributed exponentially with mean $1/\lambda_1$, regardless of the repair distribution. For n=2, relations (3.6) and (3.12) agree with equations (17) and (21) in Epstein [31].

An Analysis of the Distribution of $T_{2,n}$

In this section, the density of $T_{2,n}$, $h_{2,n}(t)$, will be investigated when repair times follow the k-Erlang distribution with parameter $\mu > 0$; i.e., when

$$\phi(s) = \int_0^\infty e^{-st} dF(t) = \left(\frac{\mu}{s + \mu}\right)^k, \quad k \geq 1 \tag{3.13}$$

First, applying the shifting property of Laplace transforms to equation (3.11) and recalling that $\lambda_i = (n-i)\lambda$, i=0,1, obtain

$$h_{2,n}(t) = \exp\{-(n-1)\lambda t\}g(t) \tag{3.14}$$

where

$$g^*(s) = \int_0^\infty e^{-st}\, g(t)dt$$

$$= h^*_{2,n}(s - (n-1)\lambda)$$

$$= \frac{n(n - 1)\lambda^2[1 - \phi(s)]}{s(s + \lambda - n\lambda\phi(s))}$$

or, using (3.13)

$$g^*(s) = C(s)/D(s) \tag{3.15}$$

where

$$\begin{aligned} C(s) &= n(n - 1)\lambda^2[(s + \mu)^k - \mu^k]/s \\ D(s) &= (s + \lambda)(s + \mu)^k - n\lambda\mu^k \end{aligned} \tag{3.16}$$

and $k \geq 1$, $n \geq 2$; $\lambda, \mu > 0$.

Note, however, that for $0 < \mu < \lambda$ and $k \geq 1$, $n \geq 2$

$$E[\text{repair time}] = k/\mu > ((n - 1)\lambda)^{-1} = E[Y]$$

In words, the expected repair time is greater than the expected time to the next failure, given a repair has just begun. Hence, for $\mu < \lambda$, the 2-out-of-n system will, "a priori," be unreliable and the interesting case is when $\mu \geq \lambda > 0$. Under this latter condition, the following theorem will be proven:

Theorem 3.1. Let μ, $\lambda > 0$ and let k and n be integers with $k \geq 1$ and $n \geq 2$. If $\mu \geq \lambda > 0$, then the k+1 zeros of $D(s) = (s + \lambda)(s + \mu)^k - n\lambda\mu^k$ are distinct.

Proof. Assume $s = r$ is a root of multiplicity $m \geq 2$ of $D(s) = 0$; then

$$\frac{d^{j-1}}{ds^{j-1}} D(s)\bigg|_{s=r} = 0, \quad j=1,2,\ldots,m;\ m \geq 2$$

Taking j=2,

$$D'(s)\bigg|_{s=r} = (r + \lambda)k(r + \mu)^{k-1} + (r + \mu)^k = 0$$

or

$$(r+\mu)^{k-1}(k(r+\lambda)+r+\mu)=0$$

But $r \neq -\mu$ since $D(-\mu) \neq 0$, and hence

$$k(r+\lambda)+r+\mu=0$$

or

$$r=-\frac{k\lambda+\mu}{k+1}$$

However,

$$D(s)\Big|_{s=-\frac{k\lambda+\mu}{k+1}}$$

$$=-\{(\mu-\lambda)^{k+1}k^{k}+n\lambda\mu^{k}(k+1)^{k+1}\}/(k+1)^{k+1}$$

$$<0, \text{ when } \mu \geq \lambda > 0$$

Therefore, $D(s)$ cannot have any root of multiplicity $m \geq 2$ when $\mu \geq \lambda > 0$.||

Consequently, if $\{r_j\}_{j=1}^{k+1}$ are the k+1 zeros of D(s) by using (3.14), (3.15), Heaviside's expansion and Theorem 3.1

$$h_{2,n}(t)=\exp\{-(n-1)\lambda t\}\cdot\sum_{j=1}^{k+1}\exp(r_j t)C(r_j)/D'(r_j) \qquad (3.17)$$

where C(s) and D(s) are defined by (3.16).

The Number of Renewals During $T_{2,n}$

Suppose that at time t=0, a renewal process begins which generates secondary failures. These secondary events are not considered serious enough to necessitate immediate repair (Gaver and Luckew [38]) and it is assumed that total failure will not be caused by them alone (e.g., an oil leak or a faulty valve). However, when failure of the 2-out-of-n system does occur, all secondary failures are also repaired before the system is restarted.

Let the independent times between secondary failures have the distribution function $B(\cdot)$. Then, conditional upon $T_{2,n}$, it follows from renewal theory that the distribution of the number of secondary failures during $T_{2,n}$, say M, is

$$P[M = i \mid T_{2,n} = t] = B_i(t) - B_{i+1}(t), \quad i \geq 0 \tag{3.18}$$

where

$$B_{i+1}(t) = \int_{0^-}^{t} B_i(t - x)\,dB(x), \quad i \geq 0 \tag{3.19}$$

and $B_0(t)$ is the unit step function at the origin.

Now, if

$$B^*(s) = \int_0^\infty e^{-st}\,dB(t)$$

and if it can be shown that

$$\mathrm{Re}(r_j) < (n - 1)\lambda, \quad 1 \leq j \leq k+1$$

it follows that

$$\int_0^\infty \exp\{-[(n - 1)\lambda - r_j]t\}B_i(t)\,dt = \frac{[B^*((n - 1)\lambda - r_j)]^i}{(n - 1)\lambda - r_j} \tag{3.20}$$

for $1 \leq j \leq k+1$, $i \geq 0$. Using (3.17), (3.18) and (3.20), one may calculate

$$G_M(z) = \sum_{i=0}^{\infty} z^i P[M = i]$$

$$= \sum_{i=0}^{\infty} z^i \int_{t=0}^{\infty} (B_i(t) - B_{i+1}(t))h_{2,n}(t)dt$$

$$= \sum_{j=1}^{k+1} \frac{C(r_j)[1-B^*((n-1)\lambda - r_j)]}{D'(r_j)((n-1)\lambda-r_j)[1-zB^*((n-1)\lambda-r_j)]} \tag{3.21}$$

where $0 \leq z \leq 1$.

Thus, in order for (3.21) to hold, there remains to show

Theorem 3.2. If $\mu \geq \lambda > 0$, $k \geq 1$, $n \geq 2$ and if $\{r_j\}_{j=1}^{k+1}$ are the k+1 zeros of $D(s) = (s + \lambda)(s + \mu)^k - n\lambda\mu^k$, then

$$\mathrm{Re}(r_j) < (n - 1)\lambda, \quad j=1,2,\ldots,k+1 \tag{3.22}$$

Proof. By Descartes's Law of Signs, $D(s)$ has precisely one positive (real) zero, say r. Note that $D(0) < 0$ and $D((n - 1)\lambda) > 0$ and hence, it must be that

$$0 < r < (n - 1)\lambda$$

and all real zeros of $D(s)$ satisfy (3.22).

Now, with $i = \sqrt{-1}$, let $z = \alpha + \beta i$ be any complex zero of $D(s)$ such that $\mathrm{Re}(z) = \alpha \geq (n - 1)\lambda$, and let

$$Q = |(z + \lambda)(z + \mu)^k|$$

Since z is a zero of $D(s)$, it follows that

$$|(z + \lambda)(z + \mu)^k| = |n\lambda\mu^k| = n\lambda\mu^k$$

or,

$$Q = |(\alpha + \lambda + \beta i)(\alpha + \mu + \beta i)^k| = n\lambda\mu^k$$

But since $|z_1 z_2| = |z_1| \cdot |z_2|$ and $|z_1^k| = |z_1|^k$ for any two complex numbers z_1, z_2, it follows that

$$Q = |\alpha + \lambda + \beta i| \cdot |\alpha + \mu + \beta i|^k$$

$$= [(\alpha + \lambda)^2 + \beta^2]^{1/2}[(\alpha + \mu)^2 + \beta^2]^{k/2} \tag{3.23}$$

However, for $\alpha \geq (n - 1)\lambda$, λ, $\mu > 0$, $k \geq 1$, $n \geq 2$ and for all real values of β,

$$[(\alpha + \lambda)^2 + \beta^2]^{1/2} \geq [(n\lambda)^2 + \beta^2]^{1/2} \geq n\lambda > 0 \tag{3.24}$$

and

$$[(\alpha + \mu)^2 + \beta^2]^{k/2} \geq [((n - 1)\lambda + \mu)^2 + \beta^2]^{k/2}$$

$$\geq ((n - 1)\lambda + \mu)^k > \mu^k > 0 \tag{3.25}$$

Combining (3.24) and (3.25) with (3.23)

$$Q = [(\alpha + \lambda)^2 + \beta^2]^{1/2}[(\alpha + \mu)^2 + \beta^2]^{k/2} > n\lambda\mu^k$$

whenever $\alpha \geq (n - 1)\lambda$. Hence, when $\alpha = \mathrm{Re}(z) \geq (n - 1)\lambda$, the complex number $z = \alpha + \beta i$ cannot be a zero of $D(s)$ and the theorem is proven.||

From (3.21), all moments of M, the number of secondary failures during $T_{2,n}$, are available by differentiating $G_M(z)$. In particular

$$E[M] = \sum_{j=1}^{k+1} \frac{C(r_j)B^*((n - 1)\lambda - r_j)}{D'(r_j)((n-1)\lambda - r_j)[1 - B^*((n-1)\lambda - r_j)]}$$

The Two-Unit Standby Redundant System

Assumptions and Definitions

The following will be assumed:

(1) The time to failure of unit 1 is a random variable which is independent of the time to failure of unit 2, and switchover time is instantaneous.

(2) At $t=0$, unit 1 is operating and unit 2 is a "cold" back-up.

(3) There is only one repairman with unlimited service capabilities. The repair time for unit 1 is a random variable which is independent of the repair time for unit 2, and the repair times and failure times are independent of each other.

Comment: Once again, to avoid cumbersome or new notation, the random variables X_i, R_i, $\tilde{R}_i$, R, I, N and M will continue to be used, even though their meaning may be different than in previous sections. The reader is cautioned to make note of this whenever these quantities are defined below.

The following notation will be used:

$\ell\{\cdot\}$ - the Laplace transform operator; i.e., $\ell\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$.

$\hat{Z}(t)$ - the number of units down at moment t ($\hat{Z}(t)$ = 0, 1 or 2).

T - the time to system failure for the two-unit standby redundant system. The density of T and its Laplace transform will be denoted by $h_T(t)$ and $h_T^*(s)$, respectively.

i - subscript denoting unit i, i=1,2.

X_i - time to failure of unit i.

$g_i(\cdot)$ - probability density function of X_i.

$G_i(\cdot)$ - distribution function of X_i.

$\overline{G}_i(\cdot) = 1 - G_i(\cdot)$

R_i - time needed to repair unit i.

$f_i(\cdot)$ - probability density function of R_i.

$F_i(\cdot)$ - distribution function of R_i.

$\overline{F}_i(\cdot) = 1 - F_i(\cdot)$

N - the number of repairs completed prior to system failure.

R - the total time spent on repair during T.

$\overline{R}_i$ - the total time spent repairing unit i during T.

I - the total time that the repairman is idle during T.

A Conditional Transform Approach

Before deriving the distribution of N, consider some typical realizations of the process $\hat{Z}(t)$ (Figures 3.2 and 3.3).

Now, let

$$p = P[R_1 < X_2] = \int_0^\infty g_2(t)F_1(t)dt = \int_0^\infty f_1(t)\overline{G}_2(t)dt$$

$$q = P[R_2 < X_1] = \int_0^\infty g_1(t)F_2(t)dt = \int_0^\infty f_2(t)\overline{G}_1(t)dt$$

and note that (see Figures 3.2 and 3.3)

$$P[N = n, N \text{ odd}] = p^{(n+1)/2}\, q^{(n-1)/2}\,(1-q), \quad n=1,3,5,\ldots \tag{3.26}$$

$$P[N = n, N \text{ even}] = (pq)^{n/2}(1 - p), \quad n=0,2,4,\ldots \tag{3.27}$$

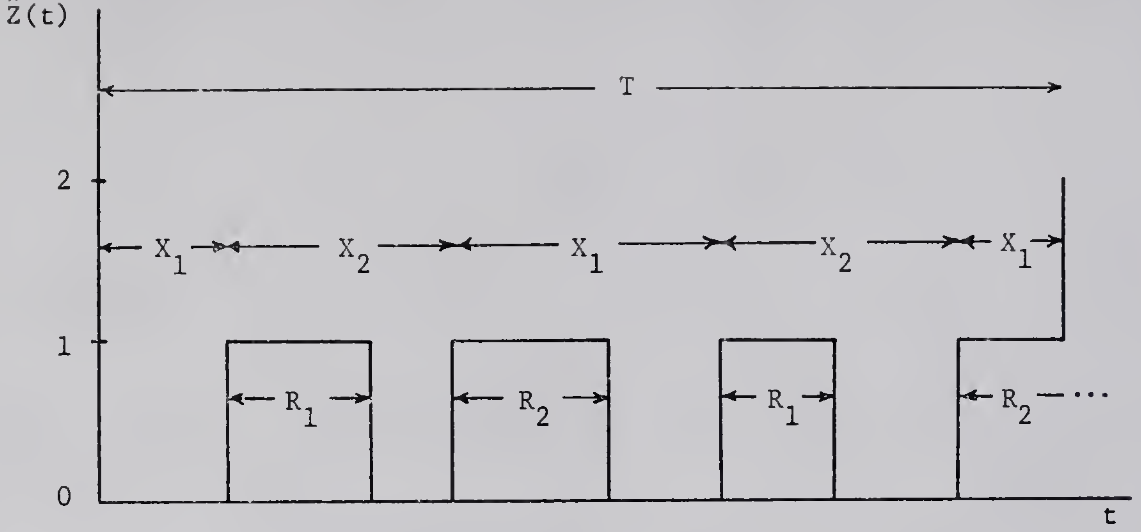

Figure 3.2. A sample function of $\hat{Z}(t)$ in terms of the random variables X_i, R_i and T, when an odd number of repairs (3) are completed.

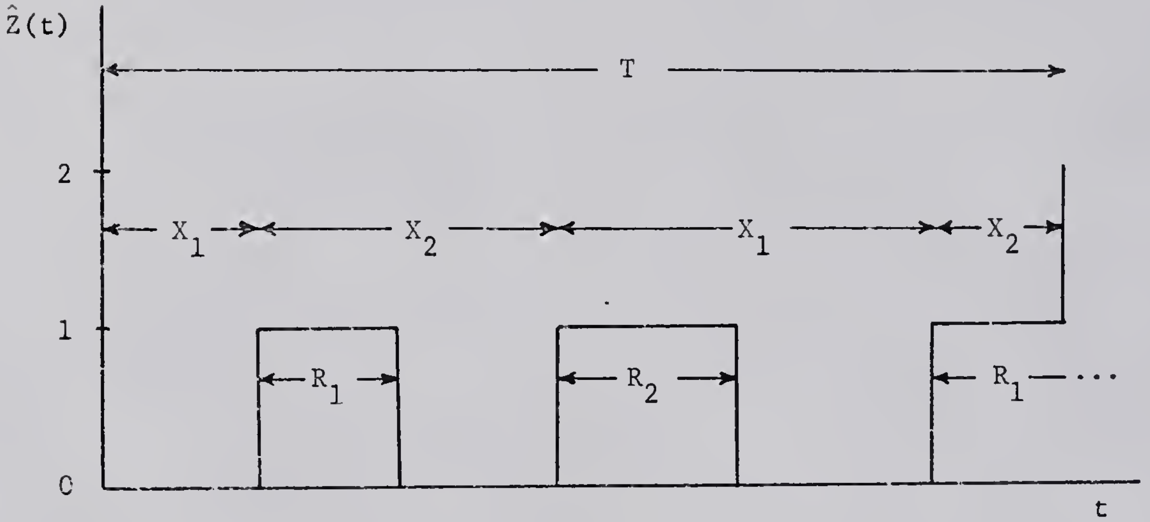

Figure 3.3. A sample function of $\hat{Z}(t)$ in terms of the random variables X_i, R_i and T, when an even number of repairs (2) are completed.

or

$$P[N = n] = \frac{1}{2}\{(-1)^{n+1} + 1\}p^{(n+1)/2}\, q^{(n-1)/2}\,(1 - q) +$$

$$+ \frac{1}{2}\{(-1)^{n} + 1\}(pq)^{n/2}(1 - p), \quad n=0,1,2,\ldots \qquad (3.28)$$

As a check, using (3.28)

$$G_N(z) = \sum_{n=0}^{\infty} z^n P[N = n]$$

$$= \frac{1 - p + p(1 - q)z}{1 - z^2 pq}, \quad 0 \le z \le 1 \qquad (3.29)$$

and from (3.29)

$$G_N(1) = 1$$

To find the transform of the distribution of T, a conditional transform approach is used:

$$E[e^{-sT}] = \sum_{n\ \mathrm{odd}} E[e^{-sT}|N = n,\ N\ \mathrm{odd}] \cdot P[N = n,\ N\ \mathrm{odd}] +$$

$$+ \sum_{n\ \mathrm{even}} E[e^{-sT}|N = n,\ N\ \mathrm{even}] \cdot P[N = n,\ N\ \mathrm{even}] \qquad (3.30)$$

For N = n and N odd

$$T = X_1 + \sum_{j=1}^{(n-1)/2} \tilde{X}_{1,j} + \sum_{j=1}^{(n+1)/2} \tilde{X}_{2,j} + X_1^*$$

where, the $\{\tilde{X}_{1,j}\}$, $j \ge 1$, are i.i.d. as a random variable $\tilde{X}_1$ with

$$dP[\tilde{X}_1 \le t] = \frac{g_1(t)F_2(t)}{q}\, dt$$

and $\{\tilde{X}_{2,j}\}$, $j \geq 1$, are i.i.d. as a random variable $\tilde{X}_2$ with

$$dP[\tilde{X}_2 \leq t] = \frac{g_2(t)F_1(t)}{p}\, dt$$

and

$$dP[X_1^* \leq t] = \frac{g_1(t)\overline{F}_2(t)}{1 - q}\, dt$$

Also, for $N = n$ and N even

$$T = X_1 + \sum_{j=1}^{n/2} \tilde{X}_{1,j} + \sum_{j=1}^{n/2} \tilde{X}_{2,j} + X_2^*$$

where

$$dP[X_2^* \leq t] = \frac{g_2(t)\overline{F}_1(t)}{1 - p}\, dt$$

Defining

$$g_1^*(s) = \ell\{g_1(t)\}$$

and combining (3.26) and (3.27) with (3.30) and the above discussion, it follows that

$$\begin{aligned} h_T^*(s) &= E[e^{-sT}] \\ &= g_1^*(s) \sum_{n \text{ odd}} \left[\left(\frac{\ell\{g_2(t)F_1(t)\}}{p} \right)^{(n+1)/2} \left(\frac{\ell\{g_1(t)F_2(t)\}}{q} \right)^{(n-1)/2} \cdot \right. \\ &\quad \left. \cdot \left(\frac{\ell\{g_1(t)\overline{F}_2(t)\}}{1 - q} \right) p^{(n+1)/2}\, q^{(n-1)/2} (1 - q) \right] + \\ &\quad + g_1^*(s) \sum_{n \text{ even}} \left[\left(\frac{\ell\{g_2(t)F_1(t)\}}{p} \right)^{n/2} \left(\frac{\ell\{g_1(t)F_2(t)\}}{q} \right)^{n/2} \cdot \right. \\ &\quad \left. \cdot \left(\frac{\ell\{g_2(t)\overline{F}_1(t)\}}{1 - p} \right) (pq)^{n/2} (1 - p) \right] \end{aligned}$$

$$= g_1^*(s) \frac{\ell\{g_2(t)\bar{F}_1(t)\} + \ell\{g_1(t)\bar{F}_2(t)\} \cdot \ell\{g_2(t)F_1(t)\}}{1 - \ell\{g_2(t)F_1(t)\} \cdot \ell\{g_1(t)F_2(t)\}} \tag{3.31}$$

Equation (3.31) agrees with equation (90) in Srinivasan [25] and equation (14) in Osaki [34].

Using a similar approach, transforms of the distributions of R, $\bar{R}_1$, $\bar{R}_2$ and I can be found. In particular, for $N = n$ and N odd

$$R = \sum_{j=1}^{(n+1)/2} \tilde{R}_{1,j} + \sum_{j=1}^{(n-1)/2} \tilde{R}_{2,j} + X_1^*$$

and for $N = n$ and N even

$$R = \sum_{j=1}^{n/2} \tilde{R}_{1,j} + \sum_{j=1}^{n/2} \tilde{R}_{2,j} + X_2^*$$

where, the $\{\tilde{R}_{1,j}\}$, $j \geq 1$, are i.i.d. as a random variable $\tilde{R}_1$ with

$$dP[\tilde{R}_1 \leq t] = \frac{f_1(t)\bar{G}_2(t)}{p}\, dt$$

and $\{\tilde{R}_{2,j}\}$, $j \geq 1$, are i.i.d. as a random variable $\tilde{R}_2$ with

$$dP[\tilde{R}_2 \leq t] = \frac{f_2(t)\bar{G}_1(t)}{q}\, dt$$

and $dP[X_j^* \leq t]$, $j = 1,2$, are as defined previously. Hence, from (3.26), (3.27) and the above discussion

$$E[e^{-sR}] = \sum_{n \text{ odd}} E[e^{-sR} | N = n, N \text{ odd}] P[N = n, N \text{ odd}] +$$

$$+ \sum_{n \text{ even}} E[e^{-sR} | N = n, N \text{ even}] P[N = n, N \text{ even}]$$

$$= \frac{\ell\{g_1(t)\overline{F}_2(t)\} \cdot \ell\{f_1(t)\overline{G}_2(t)\} + \ell\{g_2(t)\overline{F}_1(t)\}}{1 - \ell\{f_1(t)\overline{G}_2(t)\} \cdot \ell\{f_2(t)\overline{G}_1(t)\}} \qquad (3.32)$$

Similarly,

$$E[e^{-s\overline{R}_1}] = \sum_{n\ \mathrm{odd}} \left[\frac{\ell\{f_1(t)\overline{G}_2(t)\}}{p}\right]^{(n+1)/2} p^{(n+1)/2}\, q^{(n-1)/2}(1 - q) +$$

$$+ \sum_{n\ \mathrm{even}} \left[\frac{\ell\{f_1(t)\overline{G}_2(t)\}}{p}\right]^{n/2} \left[\frac{\ell\{g_2(t)\overline{F}_1(t)\}}{1 - p}\right] (pq)^{n/2}(1 - p)$$

$$= \frac{(1 - q)\ell\{f_1(t)\overline{G}_2(t)\} + \ell\{g_2(t)\overline{F}_1(t)\}}{1 - q\ell\{f_1(t)\overline{G}_2(t)\}}$$

and

$$E[e^{-s\overline{R}_2}] = \sum_{n\ \mathrm{even}} \left[\frac{\ell\{f_2(t)\overline{G}_1(t)\}}{q}\right]^{n/2} (pq)^{n/2}(1 - p) +$$

$$+ \sum_{n\ \mathrm{odd}} \left[\frac{\ell\{f_2(t)\overline{G}_1(t)\}}{q}\right]^{(n-1)/2} \left[\frac{\ell\{g_1(t)\overline{F}_2(t)\}}{1 - q}\right] \cdot$$

$$\cdot\ p^{(n+1)/2}\, q^{(n-1)/2}(1 - q)$$

$$= \frac{p\ \ell\{g_1(t)\overline{F}_2(t)\} + (1 - p)}{1 - p\ \ell\{f_2(t)\overline{G}_1(t)\}}$$

A check reveals that

$$E[e^{-sR}]\Big|_{s=0} = E[e^{-s\overline{R}_1}]\Big|_{s=0} = E[e^{-s\overline{R}_2}]\Big|_{s=0} = 1$$

Using the above approach, it is easy to calculate

$$E[e^{-s_1\overline{R}_1} e^{-s_2\overline{R}_2}] = \sum_{n\ \mathrm{odd}} E[e^{-s_1\overline{R}_1} e^{-s_2\overline{R}_2} | N = n, N \text{ odd}] P[N = n, N \text{ odd}] +$$

$$+ \sum_{n\ \mathrm{even}} E[e^{-s_1\overline{R}_1} e^{-s_2\overline{R}_2} | N = n, N \text{ even}] P[N = n, N \text{ even}]$$

and show that

$$E[e^{-s_1\overline{R}_1} e^{-s_2\overline{R}_2}]\Big|_{s_1=s_2=s} = E[e^{-s(\overline{R}_1+\overline{R}_2)}] = E[e^{-sR}]$$

as expected, since $R = \overline{R}_1 + \overline{R}_2$.

The same type of conditional transform approach is used to find the transform of the distribution of I, the idle time of the repairman during T.

Referring to Figures 3.2 and 3.3, it is clear that some conditional probability density functions (CPDF's) must be calculated; namely, the CPDF of $X_2 - R_1$ given that $X_2 > R_1$, say $g_{X_2-R_1|X_2>R_1}(\cdot)$, and the CPDF of $X_1 - R_2$ given that $X_1 > R_2$, say $g_{X_1-R_2|X_1>R_2}(\cdot)$. Using the transformation of variables technique and the definitions of p and q, it can be shown that

$$g_{X_2-R_1|X_2>R_1}(y) = \int_{x=y}^{\infty} f_1(x - y)g_2(x)dx/p, \quad 0 < y < \infty$$

and

$$g_{X_1-R_2|X_1>R_2}(y) = \int_{x=y}^{\infty} f_2(x - y)g_1(x)dx/q, \quad 0 < y < \infty$$

Letting

$$g_{21}^{*}(s) = \int_{y=0}^{\infty} e^{-sy}\left(\int_{x=y}^{\infty} f_1(x - y)g_2(x)dx\right)dy$$

and

$$g_{12}^*(s) = \int_{y=0}^{\infty} e^{-sy}\left(\int_{x=y}^{\infty} f_2(x - y)g_1(x)dx\right)dy$$

and remembering that

$$g_1^*(s) = \ell\{g_1(y)\}$$

it follows that

$$E[e^{-sI}] = g_1^*(s)\left[\sum_{n \text{ odd}} \left(\frac{g_{21}^*(s)}{p}\right)^{(n+1)/2} \left(\frac{g_{12}^*(s)}{q}\right)^{(n-1)/2} \cdot\right.$$

$$\cdot\ p^{(n+1)/2}\ q^{(n-1)/2}\ (1 - q) +$$

$$\left. + \sum_{n \text{ even}} \left(\frac{g_{21}^*(s)}{p}\right)^{n/2} \left(\frac{g_{12}^*(s)}{q}\right)^{n/2} (pq)^{n/2}(1 - p)\right]$$

$$= \frac{g_1^*(s)[(1 - q)g_{21}^*(s) + 1 - p]}{1 - g_{12}^*(s)g_{21}^*(s)} \tag{3.33}$$

Since $g_1^*(0) = 1$, $g_{21}^*(0) = p$ and $g_{12}^*(0) = q$, from (3.33)

$$E[e^{-sI}]\Big|_{s=0} = 1$$

as it should be.

Although $E[e^{-s(R+I)}]$ is not readily available, using (3.31), (3.32) and (3.33), by algebra

$$E[T] = -\frac{d}{ds} E[e^{-sT}]\Big|_{s=0}$$

$$= -\frac{d}{ds} \{E[e^{-sR}] + E[e^{-sI}]\}\Big|_{s=0}$$

$$= E[R] + E[I]$$

as expected, since T = R + I.

Although inverse transforms of $E[e^{-sT}]$, $E[e^{-sR}]$, $E[e^{-s\overline{R}_1}]$, $E[e^{-s\overline{R}_2}]$ and $E[e^{-sI}]$ are difficult to calculate for particular failure and repair distributions, moments of T, R, $\overline{R}_1$, $\overline{R}_2$ and I are available by differentiation and possibly with the aid of numerical techniques (reference Chapter 4).

An Analysis of the Distribution of T

In this section, the density of T, $h_T(t)$, will be investigated when

$$g_i(t) = \lambda e^{-\lambda t}, \quad \lambda > 0,\ t \geq 0$$

and

$$f_i(t) = \mu(\mu t)^{k-1}\exp(-\mu t)/(k-1)!, \quad \mu > 0,\ k \geq 1,\ t \geq 0$$

for i = 1,2. Defining

$$\phi(s) = \int_0^\infty e^{-st} f_i(t)dt, \quad i=1,2$$

it follows that

$$\phi(s) = \left(\frac{\mu}{s+\mu}\right)^k, \quad k \geq 1 \tag{3.34}$$

From (3.31) and the above

$$h_T^*(s) = \frac{\lambda^2[1-\phi(s+\lambda)]}{(s+\lambda)^2 - \lambda(s+\lambda)\phi(s+\lambda)} \tag{3.35}$$

As before, by the shifting property of Laplace transforms and equation (3.35)

$$h_T(t) = e^{-\lambda t}k(t) \tag{3.36}$$

where

$$k^*(s) = \int_0^\infty e^{-st}k(t)dt$$

$$= h_T^*(s - \lambda) = \frac{\lambda^2[1 - \phi(s)]}{s^2 - \lambda s\phi(s)}$$

or, using (3.34)

$$k^*(s) = c(s)/d(s) \tag{3.37}$$

where

$$c(s) = \frac{1}{s}\lambda^2\{(s + \mu)^k - \mu^k\}$$

$$d(s) = s(s + \mu)^k - \lambda\mu^k \tag{3.38}$$

The following theorem will be useful:

Theorem 3.3. If λ, $\mu > 0$, the zeros of

$$d(s) = s(s + \mu)^k - \lambda\mu^k$$

are distinct for k=1,2,....

Proof. The proof of Theorem 3.3 follows the same line of reasoning as Theorem 3.1.||

Thus, if $\{s_j\}_{j=1}^{k+1}$ are the k+1 zeros of d(s), then by (3.36), (3.37), Heaviside's expansion and Theorem 3.3

$$h_T(t) = e^{-\lambda t}\sum_{j=1}^{k+1} e^{s_j t} c(s_j)/d'(s_j) \tag{3.39}$$

where c(s) and d(s) are defined by (3.38).

The Number of Renewals During T

Again, properties of renewal theory may be used to find the generating function of the distribution of M, the number of renewals (or secondary failures) during T.

As before, let the independent times between renewals have the distribution function $B(\cdot)$. Then

$$P[M = i \mid T = t] = B_i(t) - B_{i+1}(t), \quad i \geq 0 \tag{3.40}$$

where the $B_i(\cdot)$ are defined as in equation (3.19) above.

It is asserted that:

Theorem 3.4. If $\mu, \lambda > 0$, $k \geq 1$ and if $\{s_j\}_{j=1}^{k+1}$ are the k+1 zeros[1] of $d(s) = s(s + \mu)^k - \lambda\mu^k$, then

$$\operatorname{Re}(s_j) < \lambda, \quad j=1,2,\ldots,k+1$$

Proof. The proof is almost identical to that of Theorem 3.2 and will not be repeated.||

Hence, with

$$B^*(s) = \int_0^\infty e^{-st} dB(t)$$

and using (3.39), (3.40) and Theorem 3.4, one may calculate

$$G_M(z) = \sum_{i=0}^{\infty} z^i P[M = i] = \sum_{i=0}^{\infty} z^i \int_{t=0}^{\infty} (B_i(t) - B_{i+1}(t))h_T(t)dt$$

$$= \sum_{j=1}^{k+1} \frac{c(s_j)[1 - B^*(\lambda - s_j)]}{d'(s_j)(\lambda - s_j)[1 - zB^*(\lambda - s_j)]}$$

where $0 \leq z \leq 1$.

---

[1] See Sivazlian [40] for an analysis of a special case of the polynomial d(s) which arises in inventory theory.

CHAPTER 4

APPLICATIONS

Introduction

In this chapter, applications of the theory in Chapters 2 and 3 are treated. In particular, numerical results are obtained for the 2-out-of-3 and 3-out-of-4 systems with respect to an airport limousine problem and some numerical techniques applicable to Chapters 2 and 3 are mentioned.

An Example

A city power plant system[1] or an airport limousine (also referred to as car) service are two practical situations which can be modeled as a k-out-of-n system. For instance, if a city has say three power plants of varying sizes, it is likely that if any two of them are in a failed state, the third power station will become overloaded and the entire system will fail; i.e., a 2-out-of-3 system, assuming one repair crew. A more realistic and intuitive example is that of an airport limousine service and, hence, discussions will be confined to the latter.

Consider the manager of an airport limousine service composed of three limousines. Assume that so long as at least two of the limousines are operative, airport customers will be inclined to use the limousine service. However, if at any time only one limousine is operative, customers will lose patience and choose other means of transportation (e.g., busses or taxis). Thus, the manager would be interested in knowing the mean time to failure (one limousine working) for his 2-out-of-3 system; i.e., $E[\tilde{T}_{2,3}]$ in the notation of Chapter 2. In addition, the manager might also be interested in how much an extra limousine would be worth to him. In other words, how does $E[\tilde{T}_{2,3}]$ compare with $E[\tilde{T}_{3,4}]$.

---

[1] According to Professor O. I. Elgerd, Department of Electrical Engineering, University of Florida, the k-out-of-n system is too simplified a model for power plant systems in general; however, the above discussion does provide the reader with a feeling for what k-out-of-n systems are.

Assuming limousine i has an expected time to failure of $1/\lambda_i$ weeks (1 week = 7 days), and the mean time to repair limousine i is μ_i weeks, $1 \leq i \leq 4$, calculations of

$$\tilde{W}_{2,3}(0) = E[\tilde{T}_{2,3}] \quad \text{and} \quad \tilde{W}_{3,4}(0) = E[\tilde{T}_{3,4}]$$

were made using equation (2.56) (for $\tilde{W}_{2,3}(0)$) and equations (2.63), (2.64), (2.67) and (2.69) (for $\tilde{W}_{3,4}(0)$). The results for several sets of values of $\{\lambda_i\}$ and $\{\mu_i\}$ are shown in Tables 1 and 2 for three situations, when all repair times are (1) deterministic, (2) 2-Erlang and (3) exponential.

As indicated in Table 1, for the 2-out-of-3 system exponential repair appears to be better than deterministic repair capabilities when the mean repair times are the same. This peculiarity of the 2-out-of-3 system was also mentioned by Downton [27]. However, the standard deviation of $\tilde{T}_{2,3}$ is slightly larger for exponential repair than it is for either deterministic or 2-Erlang repair (see Table 3).

A comparison of Tables 1 and 2 would aid the limousine manager in deciding the value of an extra limousine. Comparing cases 1 and 5 for example, the manager presently (Table 1) has three cars with mean

TABLE 1

$E[\tilde{T}_{2,3}]$ FOR DETERMINISTIC, 2-ERLANG AND EXPONENTIAL REPAIR

Case No.	Failure Rates and Mean Repair Times for Unit i, i=1,2,3	$E[\tilde{T}_{2,3}]$		
		Deterministic Repair	2-Erlang Repair	Exponential Repair
1	$\lambda_1 = 1/8,\ \lambda_2 = \lambda_3 = 1/4$ $\mu_1 = 1/7,\ \mu_2 = \mu_3 = 2/7$	19.39	19.80	20.19
2	$\lambda_1 = \lambda_2 = 1/8,\ \lambda_3 = 1/4$ $\mu_1 = \mu_2 = 1/7,\ \mu_3 = 2/7$	36.45	36.96	37.46
3	$\lambda_1 = \lambda_2 = \lambda_3 = 1/8$ $\mu_1 = \mu_2 = \mu_3 = 1/7$	80.00	80.67	81.33
4	$\lambda_1 = \lambda_2 = \lambda_3 = 1/4$ $\mu_1 = \mu_2 = \mu_3 = 2/7$	12.01	12.34	12.66

TABLE 2

$E[\tilde{T}_{3,4}]$ FOR DETERMINISTIC, 2-ERLANG AND EXPONENTIAL REPAIR

Case No.	Failure Rates and Mean Repair Times for Unit i, i=1,2,3,4	$E[\tilde{T}_{3,4}]$		
		Deterministic Repair	2-Erlang Repair	Exponential Repair
5	$\lambda_1 = 1/8,\ \lambda_2=\lambda_3=\lambda_4=1/4$ $\mu_1 = 1/7,\ \mu_2=\mu_3=\mu_4=2/7$	118.38	87.77	72.61
6	$\lambda_1 = 1/8,\ \lambda_2=\lambda_3=\lambda_4=1/4$ $\mu_1 = 1/7,\ \mu_2=\mu_3=2/7,$ $\mu_4 = 1$	32.88	28.41	26.32
7	$\lambda_1=\lambda_2=\lambda_3=1/8,\ \lambda_4 = 1/2$ $\mu_1=\mu_2=\mu_3=1/7,\ \mu_4 = 1$	49.53	40.18	35.68
8	$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4$ $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 2/7$	62.33	47.58	40.33

TABLE 3

THE STANDARD DEVIATION OF $\tilde{T}_{2,3}$ FOR THE FOUR CASES CONSIDERED IN TABLE 1

Case No.	Standard Deviation of $\tilde{T}_{2,3}$		
	Deterministic Repair	2-Erlang Repair	Exponential Repair
1	19.28	19.64	19.98
2	36.35	36.81	37.27
3	79.93	80.57	81.20
4	11.89	12.17	12.45

times to failure of 8 weeks, 4 weeks and 4 weeks and mean repair times are 1 day, 2 days and 2 days, respectively. For discussion purposes, a "new car" has 8 weeks and 1 day as its mean failure and repair time, and an "old car" has 4 weeks and 2 days as its mean failure and repair time. Thus, if the manager adds an old car to his original fleet of one new and two old cars, he gains with respect to the MTSF almost 100 weeks under deterministic repair, about 67 weeks under 2-Erlang repair and 50 weeks under exponential repair. If, however, after purchasing an extra old car, it suddenly starts taking a full week (on the average) to repair it (e.g., parts must be ordered), from Table 2, case 6, the MTSF shows a marked decrease, but is still better than the MTSF for his original fleet.

Even if the manager originally has a fleet of three old cars (Table 1, case 4), the purchase of an extra old car (Table 2, case 8) adds 40 weeks under deterministic repair, 35 weeks under 2-Erlang repair and 28 weeks under exponential repair, to his initial MTSF.

Although a more complete study would be necessary to draw legitimate conclusions, it is clear that the addition of one extra limousine of relatively good quality substantially increases the difference between $E[\tilde{T}_{3,4}]$ and $E[\tilde{T}_{2,3}]$.

Using (2.56), the benefit gained by having a repairman can also be seen. In particular, if $\tilde{t}_{2,n}$ is the time to system failure for the 2-out-of-n system without repair, as $\phi_i \to 0$ it follows from (2.56)

$$E[\tilde{t}_{2,n}] = \left(\sum_{i=1}^{n} \lambda_i \right)^{-1} \left(1 + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda^{(i)}} \right) , \quad n \geq 2 .$$

When n=3, values of $E[\tilde{t}_{2,3}]$ for the four cases in Table 1 are shown in Table 4. A comparison of $E[\tilde{T}_{2,3}]$ from Table 1 and $E[\tilde{t}_{2,3}]$ from Table 4 illustrates the benefit gained by having a repairman.

TABLE 4

$E[\tilde{t}_{2,3}]$ FOR THE FOUR CASES CONSIDERED IN TABLE 1

Case No.	$E[\tilde{t}_{2,3}]$
1	4.13
2	5.83
3	6.66
4	3.33

Numerical Methods Applicable to Chapters 2 and 3

Except for particular cases (e.g., when all units fail exponentially and repair time distributions are of the Erlang or gamma type), the inverse transforms of many of the results in Chapters 2 and 3 cannot be found by elementary methods. The reader is referred to Linton [41] for a numerical approach which can be used to plot the density of a random variable which is characterized by an irrational Laplace transform.

However, as long as the failure and repair distributions are expressable in closed form, moments of the random variables treated above are available. For example, the integrals necessary to find E[T] from (3.31),

$$\int_0^\infty g_2(t)F_1(t)dt \qquad \text{and} \qquad \int_0^\infty g_1(t)F_2(t)dt$$

can in some cases be computed directly using Gauss-Laguerre quadrature formulas. In any event, one may also use a transformation like $x = e^{-t}$ to transform $(0, \infty)$ to a finite interval, and then apply Simpson's rule or the Trapezoidal rule.

Hence, although moments of the random variables discussed in Chapters 2 and 3 are obtainable, finding the distribution of a random variable (or a plot of its density) from an irrational Laplace transform will depend on the structure of the case in question.

CHAPTER 5

CONCLUSIONS AND AREAS FOR FUTURE RESEARCH

Conclusions

The research reported in this dissertation has broadened both the class of systems and system characteristics which can be treated by mathematical reliability theory. Although theoretical aspects were emphasized, applications were also considered.

An investigation of methods for analyzing systems with standbys resulted in generalized theory for finding the transform of the distribution of $T_{k,n}$, the time to system failure for the k-out-of-n system. The main contribution of the research was the computation of the transform of the distribution of $T_{k,n}$ via the supplementary variable technique, under a more general set of assumptions than has been considered to date. Using the principle of regeneration, an alternative derivation of the transform of the distribution of $T_{2,n}$ was also obtained.

To investigate system characteristics which have been generally neglected in the literature, a conditional transform approach was applied to the two-unit standby redundant system and the 2-out-of-n system. Transforms of distribution were derived for the time spent on repair, the free time of the repairman and the number of repairs completed. For the case of exponential failure and Erlangian repair capabilities, the time to system failure of each system was analyzed and the generating function for the number of renewals occurring during the life of each system was found.

In the context of an airport limousine problem, numerical results were calculated for the 2-out-of-3 and 3-out-of-4 systems. The value of an extra limousine as well as the benefit gained by having a repairman were discussed. Numerical methods applicable to finding moments of random variables characterized by complex integrals were also mentioned.

Areas for Future Research

The k-out-of-n System

In Chapters 2 and 3, whenever a k-out-of-n system was treated, it was always assumed that failures were random and repair was general, but only one repairman was permitted. Hence, two obvious extensions are to allow both failures and repairs to follow general distributions and to allow several repairmen.

For the case of several repairmen and general repair distributions, the ability to analyze the system will depend, as always, on how the states are defined. If the supplementary variable approach were used, two supplementary variables would most likely be required.

When general failure and repair times are assumed, the problem of finding the distribution of T (or its Laplace transform) for even the 2-out-of-2 system has not been solved. One approach in particular is worth mentioning because of the insight it provides. For the 2-out-of-2 system with one repairman, let Z(t) be the number of units down (0, 1 or 2) at moment t. For reasons which will become clear in a moment, assume that each unit is repaired by a different repairman, and for i = 1,2, define

$$Y_i(t) = \begin{cases} 0, & \text{if unit } i \text{ is working at moment } t \\ 1, & \text{if unit } i \text{ is under repair at moment } t \end{cases}$$

Now, if T is the time to system failure, then

$$T = \min\{t; Y_1(t) = Y_2(t) = 1\} \tag{5.1}$$

A typical realization of the processes $Y_1(t)$, $Y_2(t)$ and $Z(t)$ (see Figure 5.1) helps validate the above relationship.

Although meaningful results were not obtained in this investigation, equation (5.1), or some equivalent form, could perhaps be used to find the transform of the distribution of T for this general 2-out-of-2 system.

The 2-out-of-n System and the Two-Unit Standby Redundant System

In Chapter 3, the transform of the total time spent on repair (R) was calculated for both the 2-out-of-n system and the two-unit standby redundant system using a conditional transform approach. From Figures 3.1 and 3.2, however, it is clear that computation of the appropriate stochastic integrals (Parzen [42]) would also yield the distribution of R for each system. In particular, letting

$$d_y I(t,y) = d_y P[\int_{u=0}^{T_{2,n}} Z(u)du \le y | T_{2,n} = t]$$

and

$$d_y \hat{I}(t,y) = d_y P[\int_{u=0}^{T} \hat{Z}(u)du \le y | T = t]$$

it follows that for the 2-out-of-n system

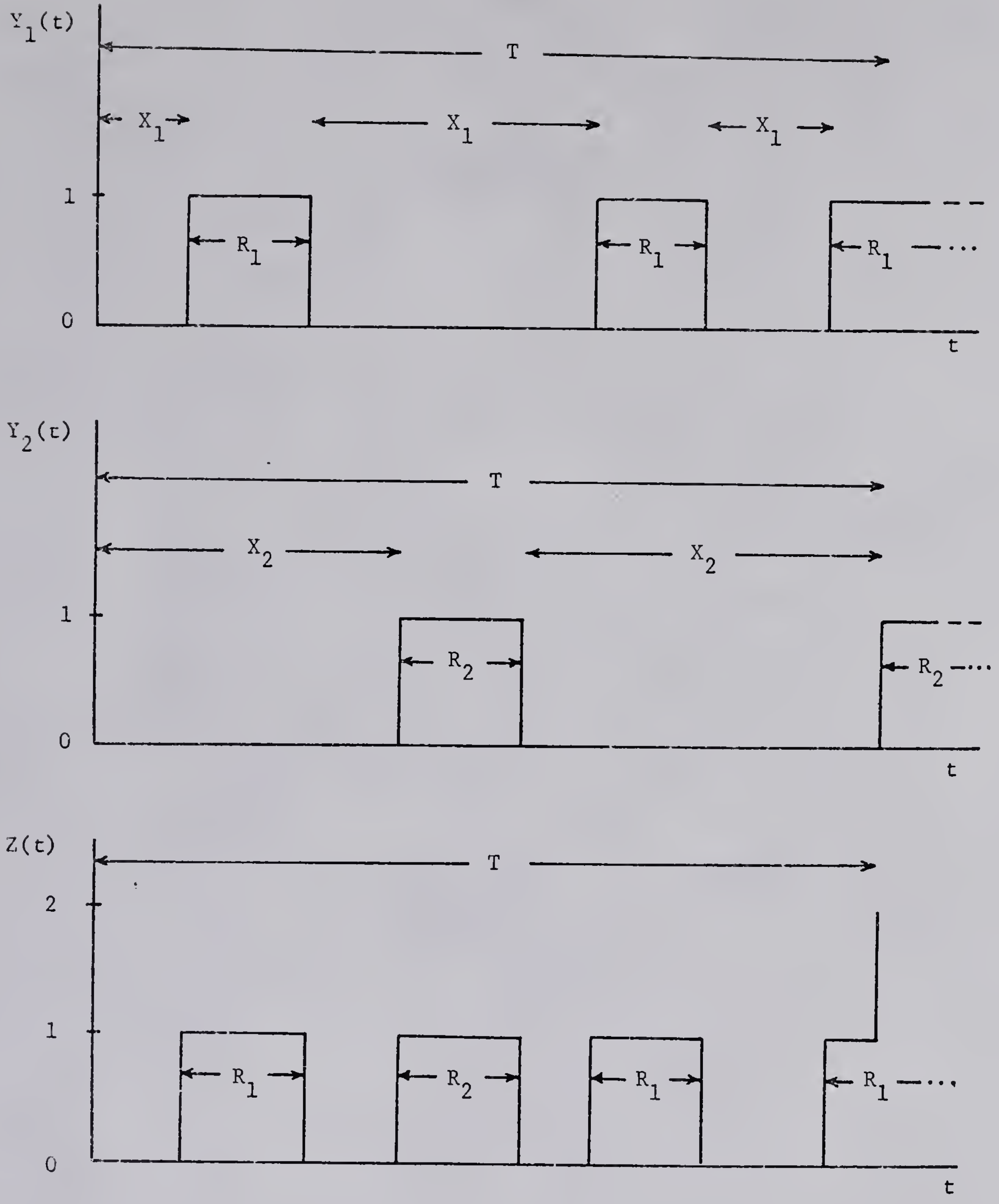

Figure 5.1. A typical realization of the processes $Y_1(t)$, $Y_2(t)$ and $Z(t)$ in terms of the random variables X_i (the time to failure of unit i), R_i (the repair time for unit i) and T.

$$\int_{y=0}^{\infty} e^{-sy}\left(\int_{t=0}^{\infty} d_y I(t,y) h_{2,n}(t)\,dt\right)dy = E[e^{-sR}]$$

and for the two-unit standby redundant system

$$\int_{y=0}^{\infty} e^{-sy}\left(\int_{t=0}^{\infty} d_y \hat{I}(t,y) h_T(t)\,dt\right)dy = E[e^{-sR}]$$

Hence, even though $d_y I(t,y)$ and $d_y \hat{I}(t,y)$ are not easily obtained, properties of the random variables

$$\int_0^{\infty} Z(u)\,du \qquad \text{and} \qquad \int_0^{\infty} \hat{Z}(u)\,du$$

are available by means of $E[e^{-sR}]$.

Although a powerful tool, the application of stochastic integrals to reliability models has not as yet been researched.

Priority Models

A third major research area concerns the use of priorities. In this dissertation, it was always assumed that units were repaired in the order in which they failed (i.e., "first-come, first-served"). By classifying each unit with a priority index, the preemptive-resume or head-of-the-line disciplines could be imposed and new results might be obtained (see Jaiswal [22] and Natarajan [28], [29]).

APPENDIX

AN APPLICATION OF THE PRINCIPLE OF REGENERATION

In the spirit of Muth [35], the principle of regeneration will be used to find the transform of the distribution of the time to system failure for the 2-out-of-n system.

Define

$\tilde{T}_{2,n}$ - The time to system failure for the 2-out-of-n system. As before, let $\tilde{h}_{2,n}(t)$ and $\tilde{h}^*_{2,n}(s)$ denote the density of $\tilde{T}_{2,n}$ and its Laplace transform, respectively.

U_i - The time to failure of unit i, $1 \leq i \leq n$. The density function of U_i is assumed to be $\lambda_i e^{-\lambda_i u_i}$, $u_i \geq 0$, $1 \leq i \leq n$.

Y_i - The time to the next failure when unit i is under repair. The density function of Y_i is $\lambda^{(i)} e^{-\lambda^{(i)} y_i}$, $y_i \geq 0$, $1 \leq i \leq n$, where

$$\lambda^{(i)} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda_j$$

X - The time elapsed until the first failure occurs. The density function of X is $\alpha e^{-\alpha x}$, $x \geq 0$, where

$$\alpha = \sum_{j=1}^{n} \lambda_j$$

D = $\tilde{T}_{2,n} - X$

R_{ij} - The time needed to perform the j-th repair of unit i, $1 \leq i \leq n$, j=1,2,.... The $\{R_{ij}\}$, $j \geq 1$, are assumed to be i.i.d. as a random variable with distribution function $F_i(\cdot)$, $1 \leq i \leq n$.

$Z(t)$ - The number of units down at moment t ($Z(t) = 0$, 1 or 2).

Assume that the failure times are independent random variables which are also independent of the repair times, and all units are operative at $t=0$.

Unless otherwise stated, density functions will be denoted by a lower case f and Laplace transforms of densities by f^*, each with an appropriate subscript for the random variable. The convolution operator will be denoted by an asterisk (*).

Before deriving $\tilde{h}^*_{2,n}(s)$, consider a typical sample function of $Z(t)$ (Figure A.1). Since $T_{2,n} = X + D$,

$$\tilde{h}_{2,n}(t) = f_X(t) * f_D(t) \tag{A.1}$$

and note that for $1 \leq i \leq n$

$$P[U_i < U_1, \ldots, U_i < U_{i-1}, U_i < U_{i+1}, \ldots, U_i < U_n] = \frac{\lambda_i}{\alpha} \tag{A.2}$$

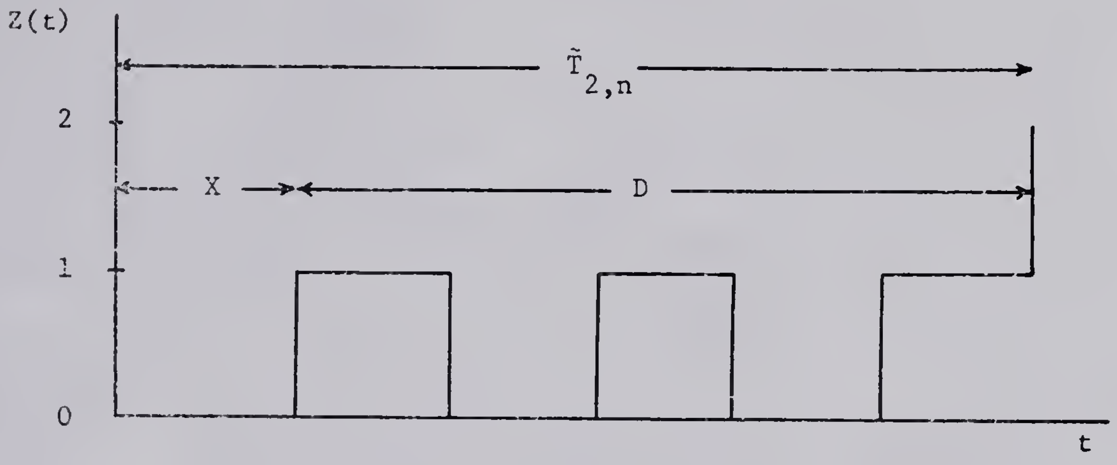

Figure A.1. A sample function of $Z(t)$ in terms of the random variables X, D and $\tilde{T}_{2,n}$.

Now, from the principle of regeneration, (A.2) and the laws of probability,

$$f_D(t) = \sum_{i=1}^{n} \left(\lambda^{(i)} e^{-\lambda^{(i)} t} (1 - F_i(t)) \frac{\lambda_i}{\alpha} \right) +$$

$$+ \int_{x=0}^{t} \left(\sum_{i=1}^{n} e^{-\lambda^{(i)} x} \, dF_i(x) \frac{\lambda_i}{\alpha} \right) \tilde{h}_{2,n}(t - x) \qquad (A.3)$$

From (A.1) and the definition of X

$$\tilde{h}^*_{2,n}(s) = \frac{\alpha}{s + \alpha} f^*_D(s) \qquad (A.4)$$

Letting

$$\phi_i(s) = \int_0^{\infty} e^{-st} \, dF_i(t), \; \psi_i(s) = \phi_i(s + \lambda^{(i)}), \; 1 \leq i \leq n$$

and transforming both sides of (A.3), it follows that

$$f^*_D(s) = \frac{1}{\alpha} \sum_{i=1}^{n} \frac{\lambda_i \lambda^{(i)} (1 - \psi_i(s))}{s + \lambda^{(i)}} +$$

$$+ \tilde{h}^*_{2,n}(s) \frac{1}{\alpha} \sum_{i=1}^{n} \lambda_i \psi_i(s) \qquad (A.5)$$

From (A.4) and (A.5)

$$\tilde{h}^*_{2,n}(s) = \frac{\displaystyle\sum_{i=1}^{n} \frac{\lambda_i \lambda^{(i)} (1 - \psi_i(s))}{s + \lambda^{(i)}}}{\displaystyle s + \sum_{i=1}^{n} \lambda_i (1 - \psi_i(s))} \qquad (A.6)$$

and equation (A.6) agrees with equation (2.55).

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BIOGRAPHICAL SKETCH

Darrell Glen Linton was born December 29, 1944, in Norfolk, Virginia. His parents moved to Baltimore, Maryland in 1945 and in June, 1962, he received his high school diploma from the Park School in Brooklandville, Maryland. From 1962 through 1966, he attended Western Maryland College at Westminster, Maryland where in June, 1966, he received the degree Bachelor of Arts with honors in mathematics. In 1966, the author's family moved to Florida, and in September, 1966, he enrolled in the Graduate School of the University of Florida at Gainesville, Florida. After a brief stay in the Department of Mathematics, he transferred to the Department of Industrial and Systems Engineering and in June, 1971, he received the degree Master of Engineering in operations research. From June, 1971, until the present he has worked as a Graduate Research Assistant and a Graduate Teaching Assistant in the Department of Industrial and Systems Engineering.

Among other honors bestowed upon the author are memberships in Kappa Mu Epsilon and Alpha Pi Mu honor societies.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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June, 1972

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